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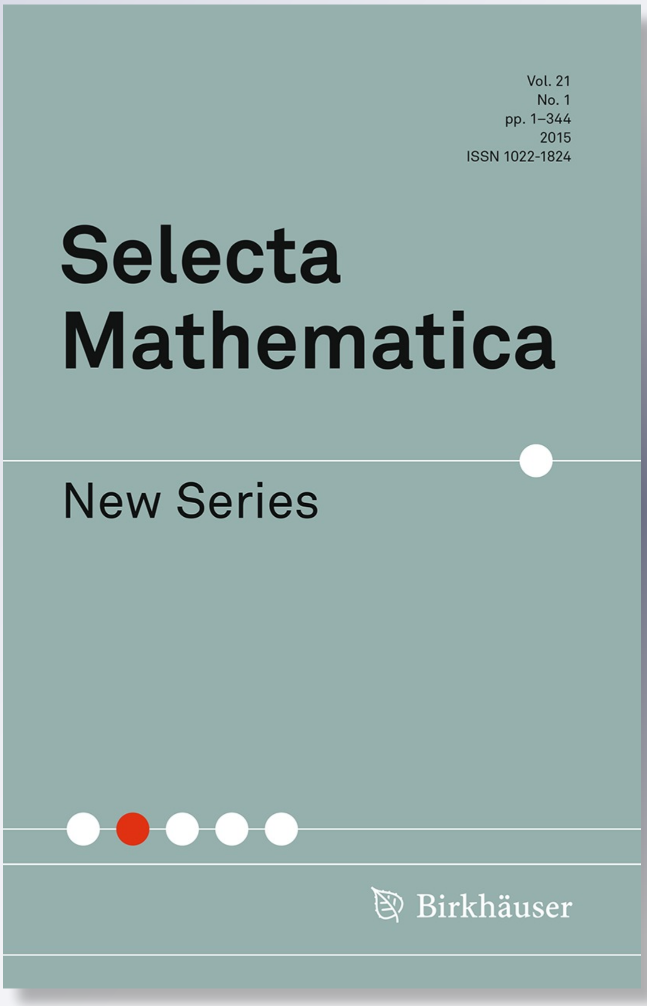

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Abstract This paper provides explicit closed formulas in terms of tautological classes for the cycle classes of the height and Artin invariant strata in families of K3 surfaces. The proof is uniform for all strata and uses a flag space as the computations in Ekedahl and van der Geer (Algebra, arithmetic and geometry, progress in mathematics, vol. 269–270, Birkhäuser, Basel, 2010) for the Ekedahl–Oort strata for families of abelian varieties, but employs a Pieri formula to determine the push down to the base space.

Keywords K3 surface · Height · Artin invariant

Mathematics Subject Classification (1991) 14C17 · 14J28 · 14H10

1 Introduction

Moduli spaces of algebraic varieties in positive characteristic possess stratifications for which there are no analogues in characteristic zero. This can make these moduli spaces more accessible than their counterparts in characteristic 0. The first example is the moduli space of elliptic curves where the distinction ordinary versus supersingular provides a stratification. This generalizes to the moduli of abelian varieties where one

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Torsten Ekedahl unexpectedly passed away on November 23, 2011.

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finds the Ekedahl–Oort stratification and the Newton polygon stratification. Besides the case of abelian varieties, where this phenomenon has attracted a lot of attention, the moduli of K3 surfaces in characteristic $p > 0$ provide a beautiful example. To define a stratification, one looks at typical characteristic p invariants, like the height and the Artin invariant. The height appears if one considers multiplication by p on the 1-dimensional formal Brauer group associated with the second étale cohomology group of a K3 surface; multiplication by p is either zero or takes the form

$$[p]\,t = a\,t^{p^h} + \text{higher order terms}$$

with $a \neq 0$ and t a local parameter. The number h is called the height and satisfies $1 \leq h \leq 10$ or we have $[p] = 0$ and then we put $h = \infty$. If $h = \infty$, the K3 surface is called supersingular (in the sense of Artin, see [2]). The loci of K3 surfaces with height $\geq h$ stratify the moduli. In the supersingular case, one finds a further invariant, the Artin invariant σ_0, by looking at the discriminant of the intersection pairing on the Néron–Severi group which is of the form $-p^{2\sigma_0}$ with $10 \geq \sigma_0 \geq 1$, and this further stratifies the smallest stratum, the supersingular locus. The generic supersingular case has $\sigma_0 = 10$, while $\sigma_0 = 1$ is the most special case. We thus find a stratification of 20 strata on the 19-dimensional moduli space, linearly ordered by inclusion.

It is the purpose of this paper to calculate closed formulas for the cycle classes of such strata in the Chow groups with rational coefficients of moduli spaces of polarized K3 surfaces. One may view such formulas as a generalization of Deuring's formula that gives the number of supersingular elliptic curves. For the strata indexed by the height of the formal Brauer group, this was done in [28] in a somewhat ad hoc manner, but for the more elusive strata parametrized by the Artin invariant, this problem remained open. It turns out that the cycle classes of the strata can be expressed in powers of the tautological class λ_1, the first Chern class of the Hodge bundle. The coefficients are complicated expressions in p, the characteristic of our ground field. The result parallels our joint work in [10] that gives such formulas for the moduli of abelian varieties.

The two invariants, the height and the Artin invariant, are of a seemingly different nature. However, they can be given a uniform description in terms of the relative position of two filtrations on the middle de Rham cohomology that refine the Hodge filtration and the conjugate filtration; this is similar to how the Ekedahl–Oort strata on the moduli of abelian varieties, originally defined in terms of group schemes, were interpreted in terms of relative position of flags on de Rham cohomology in [10,27]. These two filtrations form a so-called F-zip in the sense of [21]. The failure of transversality of these two filtrations is measured by a double coset of a Weyl group and gives rise to discrete invariants, like the height of the formal Brauer group and the Artin invariant. In contrast, in characteristic 0, the Hodge filtration and its complex conjugate are always transversal. The role of the first cohomology group for abelian varieties is replaced by the second cohomology group. We shall explain the precise relation between the discrete invariants and the relative positions of the two filtrations.

We consider here the moduli of lattice polarized K3 surfaces, i.e., K3 surfaces together with an embedding of a non-degenerate lattice in the Néron–Severi group and then consider the primitive cohomology, that is, the orthogonal complement of our non-degenerate lattice. The second de Rham cohomology of a K3 surface in characteristic p comes with two filtrations, the Hodge filtration and the conjugate filtration. We show that these two filtrations can be refined to a so-called pair of complete self-dual filtrations on the primitive cohomology which are compatible with respect to the action of Frobenius; they form a so-called complete F-zip. Such a refinement is not unique, but the relative position is controlled by an element of a Weyl group, and this element is unique.

This naturally forces us to work in a flag space $\mathcal{F}$ of the primitive part of the second de Rham cohomology over a moduli space $\mathcal{M}$ of lattice polarized K3 surfaces. Looking at the full filtrations, we find a stratification on $\mathcal{F}$ indexed by elements in a Weyl group. The strata in $\mathcal{F}$ project to strata on $\mathcal{M}$, and we are interested in formulas for the cycle classes of the strata on $\mathcal{M}$. The reason that we nevertheless insist on working on the flag space $\mathcal{F}$ is that the strata on $\mathcal{F}$ are much better behaved than on $\mathcal{M}$. Locally, on $\mathcal{F}$, one can compare the strata with the Schubert strata on the space of complete self-dual flags on an orthogonal space. The idea, already used in the analogous situation for abelian varieties in [10], is that our flag space $\mathcal{F}$ as a stratified space at a point can be identified up to the $(p-1)$st neighborhood with the flag space at an appropriate point. This provides a lot of information about local structure of our strata (dimension, Cohen-Macaulay-ness). It turns out that the strata corresponding to special elements in the Weyl group (called final elements) map in a finite surjective étale way to strata on $\mathcal{M}$.

As mentioned above, the strata on the moduli of polarized K3 surfaces (given by the height and Artin invariant) are linearly ordered. However, on the space of 1-dimensional isotropic subspaces of a n-dimensional orthogonal vector space with n even, the poset of Schubert varieties is not a total order; there are two mid-dimensional incomparable Schubert varieties (which are permuted under the orthogonal group). This points to a delicate subtlety here. In the even-dimensional case, one of the middle dimensional strata is excluded in the strata on the moduli space $\mathcal{M}$. The reason behind this is the existence of a deformation invariant, the Hodge discriminant which, depending on its value (in $\mathbf{F}_p^*/\mathbf{F}_p^{*2}$), excludes one of the mid-dimensional strata and leaves us with a linearly ordered set of strata.

This is reflected in the algebraic groups behind the scene. In the case of abelian varieties, it was shown [10] that for computing the classes the algebraic group $\mathrm{Sp}(2g)$ played an essential rôle. By analogy with the complex case, one would perhaps expect that in the case of K3 surfaces, the special orthogonal group $\mathrm{SO}(n)$ would play a similar rôle. This is almost but not quite the case, and it turns out that it is rather the full orthogonal group $\mathrm{O}(n)$ that governs the situation. When the dimension n (of the primitive part of cohomology) is odd, the distinction between $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ is not really seen (essentially as $\mathrm{O}(n)$ acts trivially on the Dynkin diagram). The case of an even n is markedly different (as this time $\mathrm{O}(n)$ acts non-trivially on the Dynkin diagram).

Apart from these complications that appear when n is even, our strategy for finding cycle class formulas is the same whether n is even or odd. Just as for the case of abelian varieties (cf., [10]), we work with a space of complete flags extending the Hodge filtration and first obtain formulas there for the classes of strata that are in bijection with the Schubert cells on the complete flag space (of $\mathrm{SO}(n)$). We then push down these formulas to the moduli space under consideration.

At this point, however, we follow a strategy which is different from that used in [10]: Instead of using formulas of Fulton and Pragacz, we shall use a Pieri type formula. This fits well with the fact that we have linearly ordered strata, but it introduces a new problem. This Pieri formula involves many different strata all of which will have to be pushed down to the moduli space. Comparing with the map from the complete flag space $\mathcal{F}\ell_n$ (of $\mathrm{SO}(n)$) to the space $\mathcal{I}(n)$ of isotropic 1-dimensional subspaces, we have the following situation. In the case of $\mathcal{F}\ell_n$, each Schubert cell of the complete flag space maps to a Schubert cell of $\mathcal{I}(n)$. For each Schubert cell of $\mathcal{I}(n)$, there is a unique Schubert cell of $\mathcal{F}\ell_n$ that maps isomorphically to it (the *final* cell in our terminology). All non-final cells map to a cell with positive dimensional fibers, and hence, their cycle classes will push down to 0. In our case, the situation is the same up to infinitesimal order $p-1$. That means that the map on a final stratum is étale, and on a non-final stratum, it is non-separable. The degree with which a final stratum maps to a stratum on our moduli space can be computed (and usually is greater than 1), and the result is analogous to the case of abelian varieties. For a non-final stratum, we can either see that its image is lower-dimensional and hence can be ignored, or we can find a factorization, called a *shuffle*, of the projection as an inseparable map of computable degree to another stratum and the projection of that latter stratum. Iterating this we either get that a stratum has lower-dimensional image or that the projection factors as an inseparable map (of computable degree) to a final stratum and the projection of the final stratum. This allows us to get a complete description of the push down of the classes coming from the Pieri formula, and thus, we get formulas for the cycle classes of the strata on the moduli space.

To give a feeling for the resulting formulas, let us consider the simple case of the moduli space $\mathcal{M}_d$ of K3 surfaces with a polarization of degree d, prime to the characteristic of the field k. One has 20 strata $\mathcal{V}_w$ parametrized by a so-called final elements $w = w_i$ with $i = 1, \ldots, 20$ in a Weyl group. The strata $\mathcal{V}_{w_j}$ for $j = 1, \ldots, 10$ are the strata of K3 surfaces whose formal Brauer group has finite height j, the stratum $\mathcal{V}_{w_{11}}$ is the supersingular stratum, and the strata indexed by w_j for $j = 12, \ldots, 20$ correspond to supersingular K3 surfaces with Artin invariant $21 - j$. The strata come with a natural scheme structure. Our result expresses the cycle classes of these strata as multiples of powers of the Hodge class $\lambda_1 = c_1(\pi_* \Omega^2_{\mathcal{X}/M}) \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}_d)$ with $\pi : \mathcal{X} \to \mathcal{M}_d$ the universal K3 surface.

Theorem 1.1 *The cycle classes of the final strata $\overline{\mathcal{V}}_w$ on the moduli space $\mathcal{M}_d$ are polynomials in λ_1 with coefficients that are 1/2 times an integral polynomial in $p \neq 2$ given by*

$$\text{i)}\quad [\overline{\mathcal{V}}_{w_k}] = (p-1)(p^2-1)\cdots(p^{k-1}-1)\lambda_1^{k-1} \quad \text{if } 1 \leq k \leq 10,$$

$$\text{ii)}\quad [\overline{\mathcal{V}}_{w_{11}}] = \frac{1}{2}(p-1)(p^2-1)\cdots(p^{10}-1)\lambda_1^{10},$$

$$\text{iii)}\quad [\overline{\mathcal{V}}_{w_{10+k}}] = \frac{1}{2}\frac{(p^{2k}-1)(p^{2(k+1)}-1)\cdots(p^{20}-1)}{(p+1)\cdots(p^{11-k}+1)}\lambda_1^{9+k} \quad \text{if } 2 \leq k \leq 10.$$

The appearance of the factor 1/2 is related to the fact that the formulas of [28, Thm 14.2 and Section 15] count the infinite height stratum doubly (cf. also [29]). The moduli space is non-complete, but the formulas still make sense on an appropriate compactification.

Such formulas can be seen as a generalization of the well-known Deuring formula for the number of isomorphism classes of supersingular elliptic curves over an algebraically closed field of characteristic p. The formulas for the height strata were already determined in [28] in a completely different way, but that approach does not generalize to the remaining strata. The above theorem corresponds to the case where $n = 2m + 1$ is odd. The more general case of moduli stacks of K3 surfaces with a marking of a non-degenerate lattice in their Néron–Severi group forces us to treat also the subtler case where n is even. We finish this paper by giving two examples that show that the even case appears quite naturally.

When dealing with K3 surfaces, we do not have to go further than $n = 21$. However, our results should also be applicable to other moduli spaces related to arithmetic subgroups of the orthogonal group, like moduli spaces of hyperkähler manifolds in positive characteristic which would give examples with larger n.

We expect that as the moduli of K3 surfaces in positive characteristic gradually become better understood, our formulas will find many applications. Here, we give two applications, one to the non-existence of supersingular elliptic K3 surfaces with a section with $\sigma_0 = 10$, see Proposition 13.1, and a similar one to Enriques surfaces.

After the first version of this paper appeared, important advances have been made concerning K3 surfaces. We mention the results of Maulik [20], Madapusi Pera [18, 19], and Charles [7] on Artin's conjecture and the Tate conjecture and results of Liedtke [17] concerning the unirationality and moduli of supersingular K3 surfaces. Artin's conjecture says that supersingular K3 surfaces (that is, of height $h = \infty$) have Picard number $\rho = 22$ (that is, are supersingular in Shioda's sense). This has been verified for $p \geq 5$; the case $p = 2$ was already done by Rudakov and Shafarevich, [26]. The papers of Madapusi Pera and Liedtke contain important information concerning moduli spaces of K3 surfaces in positive characteristic.

Conventions 1.2 Throughout this paper we assume that the characteristic p is not 2 as orthogonal groups show a different behavior in characteristic 2.

The original version of this paper was put on arXiv on April 15, 2011 (arXiv:1104.3024v1). On November 23, 2011, Torsten Ekedahl suddenly died. The second author has revised this paper trying to improve the exposition and add more explanation. The mathematical content is essentially the same as in the original preprint.

2 Combinatorics

We start with an auxiliary section on the combinatorics of the Weyl groups associated with our orthogonal groups. We distinguish the B, C, and D cases. As a general reference, the reader might use [6] or also [4].

2.1 B and C combinatorics

The Weyl group W_m^B of $\mathrm{SO}(2m+1)$ can be identified with the subgroup of S_{2m+1}, the symmetric group on $2m+1$ letters, consisting of the permutations $\sigma \in S_{2m+1}$, for which $\sigma(i) + \sigma(2m+2-i) = 2m+2$. We shall specify such a permutation by giving the images of the $1 \leq i \leq m$ as $[a_1, a_2, \ldots, a_m]$. Thus, the condition that this specify an element of W_m^B is that $a_i \notin \{a_j, m+1, 2m+2-a_j\}$ for all $i \neq j$. The elements that are reduced with respect to the set of roots obtained by removing the first root (so that the remaining roots form a root system of type B_{m-1}) are precisely those of the form $[a_1, a_2, \ldots, a_m]$ with $a_1 \neq m+1$ and $a_2, \ldots, a_m$ being an increasing sequence consisting of the first $m-1$ integers ≥ 1 which are different from a_1 and $2m+2-a_1$ (cf., [3, §3.4]). We write them as $[2m+2-a, 1, 2, 3, \ldots]$ including of course examples such as $[2, 1, 3, \ldots]$ and $[2m+1, 2, 3, \ldots]$. We shall call these elements the *final elements* of W_m^B. There are $2m$ final elements; we shall list these looking for the next largest $w(1)$ and thus denote these by $w_1 = [2m+1, 2, 3, \ldots]$, $w_2 = [2m, 1, 3, \ldots]$ (if $m \neq 1$), $\ldots$, $w_{2m} = 1$ and we sometimes write $w_\emptyset$ for w_1.

The *simple reflections* s_i for $i = 1, \ldots, m$ of W_m^B are the permutations $s_i = (i, i+1)(2m+1-i, 2m+2-i)$ for $i = 1, \ldots, m-1$ and $s_m = (m, m+2)$. We also define the *weight representation* of W_m^B on $\mathbb{Z}^m$ with basis vectors ϵ_i $(i = 1, \ldots, m)$ given by, for $\sigma \in W_m^B$,

$$\sigma(\epsilon_i) = \begin{cases} \epsilon_{\sigma(i)} & \text{if } \sigma(i) \leq m \text{ and} \\ -\epsilon_{2m+2-\sigma(i)} & \text{if } \sigma(i) > m. \end{cases}$$

We thus can view W_m^B as a reflection group of this lattice. In particular, for an element $\alpha \in \mathbb{Z}^m$ we have the reflection s_α in α with $s_\alpha(x) = x - \langle \alpha, x \rangle \alpha$; e.g. $s_i = s_{\epsilon_i}$.

For a permutation w of $\{1, 2, \ldots, n\}$, we define

$$r_w(i, j) = \#\{1 \leq a \leq i : w(a) \leq j\}$$

for $1 \leq i, j \leq n$. It is clear that a permutation is determined by this function.

The length of an element of W_m^B (in the sense of Coxeter groups) may be described in concrete terms as

$$\begin{aligned} \ell(w) = \#\{1 \leq i \leq j \leq m : w(i) > w(j)\} \\ + \#\{1 \leq i \leq j \leq m : w(i) + w(j) > 2m+2\}. \end{aligned}$$

We shall occasionally have to deal with the Weyl group W_m^C of $\mathrm{Sp}(2m)$. It has almost exactly the same description as W_m^B; in fact, it is isomorphic to it, except that it is considered as subgroup of S_{2m}:

$$W_m^C := \{\sigma \in S_{2m} : \sigma(i) + \sigma(2m+1-i) = 2m+1\};$$

a correspondence between them is given by $w \in W_m^B$ defining an element $w' \in W_m^C$ by $w' := \sigma w \sigma^{-1}$ where $\sigma(i) = i$ if $1 \leq i \leq m$ and $\sigma(i) = i - 1$ if $m+1 < i \leq 2m+1$.

The length of an element is given by

$$\begin{aligned}\ell(w) = \#\{1 \le i < j \le m : w(i) > w(j)\}\\ + \#\{1 \le i \le j \le m : w(i) + w(j) > 2m + 1\}.\end{aligned}$$

Finally, we define the *discriminant*, $\mathrm{disc}(w) \in \{+1, -1\}$, of $w \in W_m^B$ to be the sign of w as an element of S_{2m+1}. The reason for calling this homomorphism 'disc' will appear later.

2.2 D combinatorics

The Weyl group W_m^D of $\mathrm{SO}(2m)$ consists of the permutations in $\sigma \in S_{2m}$ for which $\sigma(i) + \sigma(2m + 1 - i) = 2m + 1$ and such that there is an even number of $1 \le i \le m$ for which $\sigma(i) > m$. The subgroup of S_{2m} fulfilling the same conditions except for the parity condition form a subgroup of S_{2m} which can be identified with the Weyl group W_m^C for $\mathrm{Sp}(2m)$. Hence W_m^D is a subgroup of W_m^C of index 2, and more precisely, it is the kernel of the signature homomorphism $\mathrm{sign} : W_m^C \to \pm 1$. We denote a permutation in W_m^C as $[a_1, a_2, \ldots, a_m]$. Thus, the condition that this specify an element of W_m^C is that $a_i \notin \{a_j, 2n + 1 - a_j\}$ for all $i \neq j$, and it belongs to W_m^D if also the number of a_i with $a_i > m + 1$ is even. The length of an element fulfills the formula

$$\begin{aligned}\ell(w) = \#\{1 \le i < j \le m : w(i) > w(j)\}\\ + \#\{1 \le i < j \le m : w(i) + w(j) > 2m + 1\}.\end{aligned}$$

The *simple reflections* s_i, $i = 1, \ldots, m$, of W_m^D are the permutations $s_i = (i, i + 1)(2m - i, 2m + 1 - i)$ for $i = 1, \ldots, m - 1$ and $s_m = (m - 1, m + 1)(m, m + 2)$. The simple reflections of W_m^C are the s_i, $i = 1, \ldots, m - 1$, and $s'_m = (m, m + 1)$. We also have the *weight representation* of W_m^C on $\mathbb{Z}^m$ with basis vectors ϵ_i $(i = 1, \ldots, m)$ given by

$$\sigma(\epsilon_i) = \begin{cases} \epsilon_{\sigma(i)} & \text{if } \sigma(i) \le m \text{ and} \\ -\epsilon_{2m+1-\sigma(i)} & \text{if } \sigma(i) > m. \end{cases}$$

Note that the fact that the larger group is equal to W_m^C is somewhat accidental. To us, it will rather be the Weyl group of $\mathrm{O}(2m)$ (as opposed to the Weyl group of $\mathrm{SO}(2m)$) or, equivalently, as the group generated by W_m^D and the non-trivial graph automorphism of D_m (which in the D_4 case is the one permuting the last two vertices). From the latter point of view s'_m gives a non-trivial graph automorphism, indeed it commutes with s_i, $1 \le i < m - 1$ and conjugation by it permutes s_{m-1} and s_m. To emphasize this point of view, we shall, when relevant, write the supergroup W_m^C as $W_m^{\prime D}$. In this context, we need a definition of length on $W_m^{\prime D}$ that mimics the length of W_m^D (rather than that of W_m^C) and which we shall therefore denote ℓ_D:

$$\ell_D(w) = \#\{1 \le i < j \le m : w(i) > w(j)\}$$
$$+\#\{1 \le i < j \le m : w(i) + w(j) > 2m + 1\}$$

It has the property that its restriction to W_m^D equals its natural length and that $\ell_D(ws'_m) = \ell_D(w)$.

The elements that are reduced with respect to the set of roots of D_m obtained by removing the first root (so that the remaining roots form a root system of type D_{m-1}) are precisely those of the form $[a_1, a_2, \ldots, a_m]$ with $(a_2, \ldots, a_m)$ being the lexicographically smallest sequence of integers making $[a_1, a_2, \ldots, a_m]$ an element of W_m^D. We list these by looking for the next largest $w(1)$ and thus have $w_1 := [2m, 2, 3, \ldots, m-1, m+1]$, encountering of course examples such as $[2, 1, 3, \ldots, m]$ and $[m + 1, 1, 2, \ldots, m - 1, m + 2]$. We shall call these elements the *final elements* of W_m^D. There are $2m$ final elements. The longest one is w_1 which we shall also denote $w_\emptyset$. It has the reduced expression $s_1s_2 \cdots s_{m-2}s_{m-1}s_ms_{m-2} \cdots s_1$. We also put $w'_k := w_ks'_m$, and we shall call these the *twisted final elements* with the alternative notation $w'_\emptyset$ for w'_1.

3 Flag spaces

Central in this paper are filtrations on the second the Rham cohomology of a K3 surface or on a primitive part of that. The intersection form makes this cohomology space into a quadratic space. In this auxiliary section, we shall be interested in flags in a finite-dimensional orthogonal or symplectic space, and we start by recalling some well-known facts. We refer to [3] or [6].

Let thus V be an n-dimensional vector space over a field $\mathbf{k}$ provided with a non-degenerate quadratic or symplectic form $\langle -, -\rangle$. A flag $(0) = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r$ of subspaces of V is called *isotropic* if the restriction of the form to V_r is zero. We say that the flag is *maximal* if $r = k := [n/2]$ (and hence $\dim(V_i) = i$). We can extend a maximal flag to a *self-dual complete flag* by putting $V_j = V_{n-j}^{\perp}$. The group $\mathrm{SO}(n)$ does not always acts transitively on complete flags. Two flags $V_\bullet$ and $V'_\bullet$ are in the same orbit under conjugation by $\mathrm{SO}(n)$ precisely when $\dim(V_k \cap V'_k) \equiv k \bmod 2$.

Now, given a complete flag $V_\bullet$ we may construct another complete flag $V'_\bullet$ as follows: We let $V'_i = V_i$ for $i \ne k$, and then, let V'_k be the unique maximal totally isotropic subspace containing V_{k-1} and being contained in V_{k+1} that is distinct from V_k. As $V_k \cap V'_k = V_{k-1}$ we see that $V_\bullet$ and $V'_\bullet$ are not conjugate under $\mathrm{SO}(n)$. We shall call $V'_\bullet$ the *flip* of $V_\bullet$.

When the space is symplectic or n is odd, complete flags correspond precisely to Borel subgroups of the symplectic or special orthogonal group; one associates to a flag its stabiliser. The orthogonal even case is different, however. To us, the main difference is the fact that $\mathrm{SO}(n)$ does not act transitively on complete flags.

This leads us to introduce the notion of *self-dual almost complete flag* (when $n = 2k$) which is specified by an isotropic flag $(0) = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1}$ where $\dim V_i = i$ (and hence extended to a larger flag by putting $V_j = V_{n-j}^{\perp}$ for $k + 1 \le j \le n$). If we let $\mathcal{F}_n$ be the space of almost complete flags and $\mathcal{F}'_n$ the space of complete flags, then the forgetful map $\mathcal{F}'_n \to \mathcal{F}_n$ is an étale double cover whose associated involution map $\mathcal{F}'_n \to \mathcal{F}'_n$ is given by the flip. Furthermore, $\mathrm{SO}(n)$ acts transitively on $\mathcal{F}_n$ with stabilisers the Borel group of it. On the other hand, $\mathrm{O}(n)$ acts transitively both on $\mathcal{F}_n$ and $\mathcal{F}'_n$. The stabilisers for the action on $\mathcal{F}_n$ are subgroups of $\mathrm{O}(n)$ whose intersection with $\mathrm{SO}(n)$ are Borel subgroups and which map surjectively onto $\mathrm{O}(n)/\mathrm{SO}(n)$, whereas the stabilisers on $\mathcal{F}'_n$ are the Borel subgroups of $\mathrm{SO}(n)$.

More generally, if we have an orthogonal vector bundle $\mathcal{E} \to X$ of constant rank $n = 2k$, then we have the bundle of almost complete flags $\mathcal{F}(\mathcal{E}) \to X$ and complete flags $\mathcal{F}'(\mathcal{E}) \to X$ and an étale double cover $\mathcal{F}'(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$. This double cover is actually the pullback of a double cover of X: It can be obtained by considering the quadric Y in the $\mathbb{P}^1$-bundle $\mathbb{P}(V_{k+1}/V_{k-1})$ defined by the orthogonal form; it defines a double cover, the *discriminant double cover*, $\pi : Y \to X$. Thus, we get a morphism $\mathcal{F}'(\mathcal{E}) \to Y$ which fits into a cartesian diagram

$$\begin{tikzcd}
\mathcal{F}'(\mathcal{E}) \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathcal{F}(\mathcal{E}) \arrow[r] & X.
\end{tikzcd}$$

The special properties of the even orthogonal case are the reason for the relevance of the group $W_m^{\prime D}$ as the following proposition shows. It gives representatives for the orbits of pairs of flags.

It is well known that the relative position of two flags can be measured by an element of a Weyl group. We spell out the result for the cases that interest us.

Proposition 3.1 *1) Let $e_1, \ldots, e_{2m}$ be the standard basis of a symplectic space with $\langle e_i, e_j\rangle = \delta_{i,2m+1-j}$ for $j \geq i$. The orbits of the action of $\mathrm{Sp}(2m)$ on pairs of totally isotropic complete flags in $2m$-dimensional space are in bijection with the elements of the Weyl group W_m^C. The element $w \in W_m^C$ corresponds to the orbit of $((\sum_{j\leq i} \mathbf{k}e_j), (\sum_{j\leq i} \mathbf{k}e_{w^{-1}(j)}))$.*

2) Let $e_1, \ldots, e_{2m+1}$ be the standard basis of an orthogonal space with $\langle e_i, e_j\rangle = \delta_{i,2m+2-j}$. The orbits of the action of $\mathrm{SO}(2m+1)$ on pairs of totally isotropic complete flags in $2m+1$-dimensional space are in bijection with the elements of the Weyl group W_m^B. The element $w \in W_m^B$ corresponds to the orbit of $((\sum_{j\leq i} \mathbf{k}e_j), (\sum_{j\leq i} \mathbf{k}e_{w^{-1}(j)}))$.

3) Let $e_1, \ldots, e_{2m}$ be the standard basis of an orthogonal space with $\langle e_i, e_j\rangle = \delta_{i,2m+1-j}$. The orbits of the action of $\mathrm{SO}(2m)$ on pairs of totally isotropic complete flags in $2m$-dimensional space are in bijection with the elements of the group $W_m^{\prime D}$. An element w in $W_m^{\prime D}$ corresponds to the orbit of $((\sum_{j\leq i} \mathbf{k}e_j), (\sum_{j\leq i} \mathbf{k}e_{w^{-1}(j)}))$. If $(F_\bullet, D_\bullet)$ lies in the orbit corresponding to w, then $\mathrm{disc}(w) = (-1)^d$, where $d = \dim(E_m \cap D_m)$. Flipping the first flag changes the type from w to ws'_m and flipping the second changes it from w to $s'_m w$.

Proof The first and second part is of course well known, the third part perhaps less so but in any case is just as easy to prove. □

We shall say that a basis such as in the proposition is *adapted* to the two flags. We shall also say that two complete flags are in *relative position* w for w in W_m^C, W_m^B

or $W_m'^D$, respectively if they belong to the orbit above associated with w. Note that in the B and C cases, we are dealing with orbits of G (which equals $\mathrm{SO}(2m+1)$, resp. $\mathrm{Sp}(2m)$) on the product of flag spaces $G/B \times G/B$, and we are dealing with the well-known bijection between such orbits. In the even orthogonal case (and when $\mathbf{k} = \overline{\mathbf{k}}$), flags are in bijection with $\mathrm{O}(2m)/B$, where B is a Borel group of $\mathrm{SO}(2m)$ (the stabiliser of a fixed flag) where of course $\mathrm{O}(2m)/B$ has two components. Orbits under $\mathrm{O}(2m)$ of pairs of flags are then in bijection with $W_m'^D$. We may, however, reduce ourselves at will to just the action of $\mathrm{SO}(2m)$ on $\mathrm{SO}(2m)/B$. Indeed, $V_\bullet$ and $U_\bullet$ are in relative position w precisely when $V_\bullet$ and $U'_\bullet$ are in relative position ws'_m where $U'_\bullet$ is the flip of $U_\bullet$.

All this relativizes to the situation of a symplectic or orthogonal vector bundle $\mathcal{V}$ of rank n over a base S (in which 2 is invertible). We can then construct the *flag bundle* $\mathcal{F}\ell(\mathcal{V})$ of complete self-dual flags in $\mathcal{V}$. In the even orthogonal, this factors as above through the *discriminant cover* $\mathcal{D}_{\mathcal{V}}$. The involution associated with the discriminant cover extends to an involution of $\mathcal{F}\ell(\mathcal{V})$ taking a flag to its flip. The same terminology will be used for partial flags that contain a middle dimensional member.

For later purposes, the pointwise definition of flags to be in relative position w does not suffice. We recall from [10] the scheme theoretic definition: If we have two flags over an affine scheme Y, we have two sections $s, t : Y \to T$, where T is a G/B-bundle with structure group G for a semi-simple group G and a Borel group B. Then, for any element w of the Weyl group of G, we define a (locally) closed subscheme $\mathcal{U}_w$ (resp. $\overline{\mathcal{U}}_w$) of Y in the following way. We choose locally (possibly in the étale topology) a trivialization of T for which t is a constant section. Then, s corresponds to a map $Y \to G/B$, and we let $\mathcal{U}_w$ (resp. $\overline{\mathcal{U}}_w$) be the inverse image of the B-orbit BwB (resp. of its closure in G/B). Another trivialization will differ by a map $Y \to B$; as BwB and its closure are B-invariant, these definitions are independent of the chosen trivializations and hence give global subschemes on Y. If s and t have the property that $Y = \mathcal{U}_w$, then we shall say that s and t are in *relative position* w, and if $Y = \overline{\mathcal{U}}_w$, we shall say that s and t are in *relative position* $\leq w$.

4 *F*-zips

The Hodge filtration and the conjugate filtration on the second de Rham cohomology (or a primitive part of that) of a K3 surface form a so-called orthogonal F-zip. In this auxiliary section, we introduce the stack of flagged F-zips and certain substacks of it.

Recall (cf., [21]) that an orthogonal or symplectic F-zip is a tuple $(M, C^\bullet, D_\bullet, \varphi_\bullet)$, where M is an orthogonal or symplectic vector bundle over a base of positive characteristic, $0 = C^r \subseteq C^{r-1} \subseteq \cdots \subseteq C^0 = M$ and $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_r = M$ are self-dual (not necessarily complete) flags on M and $\varphi_\bullet$ a collection of isomorphisms $\varphi_i : F^*(C^i/C^{i+1}) \to D_{i+1}/D_i$ compatible with the isomorphisms $C^{i+1}/C^i \xrightarrow{\sim} (C^{r-i}/C^{r-i-1})^*$ and $D_{i+1}/D_i \xrightarrow{\sim} (D_{r-i}/D_{r-i-1})^*$ induced by the pairing. If the rank of D_i has the constant value n_i we say that the F-zip is of *dimension type* $\underline{n} = (n_r, n_{r-1}, \ldots, n_0)$. A *flagged F-zip* is an F-zip together with a complete self-dual flag $0 = E_0 \subset E_1 \subset \cdots \subset E_n = M$ with $C^i = E_{n_i}$ where $n_i := \mathrm{rk}(C^i)$ (and $\mathrm{rk}(E_i) = i$). We can use $\varphi_\bullet$ to extend the D flag to a complete self-dual flag $G_\bullet$ by the condition that $D_{i+1} \subseteq G_{j'} \subseteq D_i$ when $C^i \subseteq E_j \subseteq C^{i+1}$, where $j' - j = n_{m-i-1} - n_i$, m being the rank of M, and $G_{j'}/D_{i+1} = \varphi_i(F^*(E_j/C^i))$.

We now want to introduce the stack of flagged F-zips. Starting from the algebraic stacks $\mathrm{BO}(m)$ and $\mathrm{BSp}(m)$ of orthogonal respectively symplectic vector bundles of rank m (over $\mathbf{Z}/p$), one builds the algebraic stack $\mathcal{ZF}$ of flagged F-zips (with $\mathrm{rk}\, D^i = n_i$ and just as we have fixed these ranks we fix whether or not we have a symplectic or orthogonal bundle). If we only have an incomplete (but still self-dual) flag extending $C^\bullet$, we shall speak of a *partially flagged F-zip* and we can do the same construction getting a partial flag extending $D_\bullet$. A partially flagged F-zip is *stable* if for every i and every k, we have that $D_i \cap C^k + C^{k+1}$ is equal to some C^j (or in relevant cases the middle part of the flip of $C^\bullet$), where i and k are chosen so that D_i, C^k and C^{k+1} are defined; here, relevant means whenever the dimension is even and the partial flag involves the middle part. The canonical map of $\mathcal{ZF}$ to the stack of orthogonal F-zips $\mathcal{Z}$ is relatively representable, so $\mathcal{ZF}$ is an algebraic stack.

If the rank of the flagged orthogonal F-zip (M, E, G) is even, we can replace both E and G by their flips, and it is easy to see that we get a new flagged F-zip which will be called the *flip* of the flagged F-zip.

Fixing the rank, n, of M and the flavor (symplectic or orthogonal) the relative position of the flags $C^\bullet$ and $D_\bullet$ is, by Proposition 3.1, described by an element w of $W^C_{n/2}$, $W^B_{(n-1)/2}$, and W'^D_n when the F-zip is symplectic, orthogonal with n odd, and orthogonal with n even, respectively. We call the F-zip of *type* w in this case.

This defines locally closed substacks $\mathcal{ZF}_w$ of $\mathcal{ZF}$ consisting of those flags (of fixed flavor and ranks $\underline{n}$) of relative position w, i.e., type w. We now also fix the sequence $\underline{n} = (0 = n_0 < n_1 < \cdots < n_r = n)$ where we demand that $\mathrm{rk}(D_i) = n_i$ (in particular self-duality forces $n_{i+1} - n_i = n_{r-i} - n_{r-i-1}$) and let $w_\emptyset$ be the element of the appropriate Weyl group (as subgroup of S_n) that takes the first n_0 integers to the last n_0 (in order), the next $n_1 - n_0$ integers to the $n_1 - n_0$ last (still in order) and so on (that is, it sends the interval $[n_i + 1, n_{i+1}]$ to $[n + 1 - n_{i+1}, n - n_i]$ preserving order).

It will be useful to have an explicit scheme with a flagged F-zip of type w over it that lies faithfully flat over $\mathcal{ZF}_w$.

Construction Given an element w in the appropriate group we define a flagged F-zip over the affine scheme $\mathrm{Spec}(\mathbf{F}_p[x_{ij}]_{1\le i<j\le n}/I)$ as follows, where generators of the ideal I are specified below:

- e_i, $i = 1, \ldots, n$, is a basis for M with $\langle e_i, e_j\rangle = \delta_{i,n+1-j}$ for $i \le j$.
- C^i has $e_1, \ldots, e_i$ as a basis and D_i has $e_{w^{-1}(1)}, \ldots, e_{w^{-1}(i)}$ as a basis.
- For $n_k < i \le n_{k+1}$ we have $\varphi_k(e_i) = e_{w^{-1}w_\emptyset(i)} + \sum_{w_\emptyset w(j)<i} x_{ij}e_j \bmod D_{n_{r-k-1}}$.
- The matrix $\mathrm{Id}_n + (x_{ij})$ is symplectic or orthogonal, respectively with respect to the scalar product of the basis $e_1, \ldots, e_n$.
- $x_{ij} = 0$ unless $i \prec j$, where $i \prec j$ precisely when $w^{-1}(i) > w^{-1}(j)$ for $n_\ell < j < i \le n_{\ell+1}$ for some ℓ with $n_\ell < n/2$.

When n is odd, then we can define another flagged F-zip over Spec $(\mathbf{F}_p[x_{ij}]_{1\le i<j\le n}/I)$ with the same definition except that we let

$$\varphi(e_{(n+1)/2}) = -e_{(n+1)/2} + \sum_{w_\emptyset w(j)<(n+1)/2} x_{(n+1)/2,j}e_j$$

instead of $e_{(n+1)/2} + \sum_{w_\emptyset w(j)<(n+1)/2} x_{(n+1)/2,j}e_j$. Therefore, we define Y_w as $\mathrm{Spec}(\mathbf{F}_p[x_{ij}]_{1\le i<j\le n}/I)$ if n is even and the disjoint union of two copies of it when n is odd. In both cases there is a flagged F-zip $\mathcal{F}_n$ over Y_w. When n is even it is the one constructed above. When n is odd we have the one with $\varphi(e_{(n+1)/2}) = e_{(n+1)/2} + \cdots$ on one copy and the one with $\varphi(e_{(n+1)/2}) = -e_{(n+1)/2} + \cdots$ on the other (the flip). By construction the two flags are everywhere of type w. This gives us a map $Y_w \to \mathcal{ZF}_w$.

Proposition 4.1 *The map $Y_w \to \mathcal{ZF}_w$ is faithfully flat.*

Proof By assumption there is a frame space $\mathcal{FF}_w \to \mathcal{ZF}_w$ of bases of a versal flagged F-zip on $\mathcal{ZF}_w$ adapted to the two flags. It is a group torsor and since the group scheme is flat also faithfully flat so that it is enough to show that the induced map $Y_w \to \mathcal{FF}_w$ is faithfully flat. There are functions x_{ij} for $n_\ell < j \le i \le n_{\ell+1}$ such that $\varphi_\ell(e_i) = x_{ii}e_{w^{-1}w_\emptyset(i)} + \sum_{w_\emptyset w(j)<i} x_{ij}e_j \bmod D_\ell$, where $x_{ii} \neq 0$. Let T consist of the diagonal automorphisms $e_i \mapsto t_ie_i$, where $t_i \cdot t_{n+1-i} = 1$ and $t_{(n+1)/2} = 1$ if n is odd. It transforms a φ into another F-zip with $x_{ii} = t^{-1}_{w^{-1}w_\emptyset(i)}t_i^p$. As the endomorphism of T given by $(t_i) \mapsto (t^{-1}_{w^{-1}w_\emptyset(i)}t_i^p)$ is separable (inducing $-w^{-1}w_\emptyset(i)$ on the Lie algebra), we get that the map from the substack of $\mathcal{FF}_w$ with $x_{ii} = 1$ for $i \neq (n+1)/2$ and $x_{ii} = \pm 1$ if $i = (n+1)/2$ to $\mathcal{FF}_w$ is an equivalence and hence we may restrict to it.

It remains to show that we may remove the x_{ij} with $n_\ell < j < i \le n_{\ell+1}$ and $w^{-1}(j) < w^{-1}(i)$. Under those assumptions, the change of basis $e'_i = e_i + \lambda e_j$, $e'_{\bar{j}} = e_{\bar{j}} - \lambda e_{\bar{i}}$, with $\bar{x} = n + 1 - x$, preserves both flags. We now have

$$\varphi_k(e'_i) = \varphi_k(e_i + \lambda e_j) = e_{w^{-1}w_\emptyset(i)} + \sum_{w_\emptyset w(k)<i} x_{ik}e_k + \lambda^p\left(e_{w^{-1}w_\emptyset(j)} + \sum_{w_\emptyset w(\ell)<j} x_{i\ell}e_\ell\right)$$

and we try to choose λ such that the coefficient in front of $e_{w^{-1}w_\emptyset(j)}$ is equal to zero. This gives a monic equation in λ with λ^p as top term and hence defines a surjective finite flat covering. We can repeat this construction in a way so that we take the largest i and j first in order for subsequent operations not to reintroduce nonzero coefficients in positions where they have been removed. At the end, we get the chosen F-zip on Y_w which shows fully faithful flatness as each step is fully faithfully flat. □

5 The Hodge discriminant

As alluded to in the Introduction, there is a subtle invariant of the cohomology that prevents one of the middle dimensional strata to turn up in the stratification of our moduli spaces. In this section, we study this invariant.

We begin now by introducing a discriminant which is a Hodge theoretic description of Ogus' crystalline discriminant (defined under slightly more general circumstances). For that, we need the determinant of a complex in the sense of Mumford and Knudsen (cf., [15]). Recall that in order to get the signs right, the determinant is a *graded line bundle*, i.e., a pair $(\ell, \mathcal{L})$ where ℓ is an integer and $\mathcal{L}$ a line bundle. This is then used in the commutativity isomorphism $L \otimes M \xrightarrow{\sim} M \otimes L$ where the sign $(-1)^{\ell m}$ is used, where ℓ and m are the degrees of L and M, respectively. The coherence conditions proved in [15] then show that we get an unambiguous isomorphism between tensor products of the same graded line bundles in different order.

We shall need one property of the determinant beyond those of [15, Def. 4]: If, for a perfect complex C of $\mathcal{O}_S$-modules, S a scheme, we let $C^* := \mathrm{RHom}_S(C, \mathcal{O}_S)$, then we have a canonical isomorphism $\rho_C : \det(C)^* \cong \det(C^*)$ functorial for quasi-isomorphisms. Indeed, this can be shown by verifying that $C \mapsto (\det(C^*))^*$ verifies the conditions of [15, Def. 4], and hence, by [15, Thm. 2], it is (canonically) isomorphic to $C \mapsto \det C$. (It can also be done by direct verification.) In any case, note that we define the dual of a graded line bundle as $(\ell, \mathcal{L})^* = (-\ell, \mathcal{L}^*)$ and that we identify $(\mathcal{L} \otimes \mathcal{M})^* = \mathcal{M}^* \otimes \mathcal{L}^*$ by the pairing

$$\begin{aligned}(\mathcal{L} \otimes \mathcal{M}) \otimes (\mathcal{M}^* \otimes \mathcal{L}^*) = \mathcal{L} \otimes (\mathcal{M} \otimes \mathcal{M}^*) \otimes \mathcal{L}^* &\xrightarrow{\mathrm{id} \otimes \mathrm{ev}_{\mathcal{M}} \otimes \mathrm{id}} \mathcal{L} \otimes \mathcal{O}_S \otimes \mathcal{L}^* \\ &= \mathcal{L} \otimes \mathcal{L}^* \to \mathcal{O}_S,\end{aligned}$$

where we have used the above sign rule for the permutation. The unicity (as well as direct computation) also gives that we have a commutative diagram

$$\begin{tikzcd}
\det(C^*)^* \arrow[r, "\rho_C^*"] \arrow[d, "\rho_{C^*}"] & \det(C)^{**} \\
\det(C^{**}) & \det(C), \arrow[l, "\det(\mathrm{ev}_C)"'] \arrow[u, "\mathrm{ev}_{\det(C)}"']
\end{tikzcd} \tag{5.1}$$

where $\mathrm{ev}_C : C \to C^{**}$ is the evaluation map (and similarly for $\mathrm{ev}_{\det(C)}$). If $\to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to$ is a distinguished triangle of perfect complexes, we have a distinguished triangle $\to \mathcal{G}^* \to \mathcal{F}^* \to \mathcal{E}^* \to$ and the resulting identification

$$(\det \mathcal{E} \otimes \det \mathcal{G})^* = \det(\mathcal{F})^* = \det(\mathcal{F}^*) = \det(\mathcal{G}^*) \otimes \det(\mathcal{E}^*) = \det(\mathcal{G})^* \otimes \det(\mathcal{E})^*$$

is then a special case of the above identification.

Now, let $\pi : X \to S$ be a smooth and proper map of schemes of pure relative dimension n over a base S of positive characteristic $p \neq 2$. Let $\mathcal{L}$ be the determinant of $R\pi_*\Omega^\bullet_{X/S}$ (which exists as $R\pi_*\Omega^\bullet_{X/S}$ is a perfect complex). By Poincaré duality, we have a canonical isomorphism $(R\pi_*\Omega^\bullet_{X/S})^* \xrightarrow{\sim} R\pi_*\Omega^\bullet_{X/S}[-2n]$ which upon taking determinants gives an isomorphism

$$\mathcal{L}^* \xrightarrow{\rho} \det(R\pi_*\Omega^\bullet_{X/S}[-2n]) \xrightarrow{\sim} \mathcal{L}^{(-1)^{2n}} = \mathcal{L}.$$

Now, Poincaré duality gives a symmetric pairing, and by (5.1), this gives a perfect symmetric pairing $\mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_S$. On the other hand, the naive truncations[1] of the de Rham complex give rise to distinguished triangles

$$\to R\pi_*\Omega^{\geq i+1}_{X/S} \to R\pi_*\Omega^{\geq i}_{X/S} \to R\pi_*\Omega^i_{X/S}[-i] \to .$$

Taking determinants, we get (cf., [15, Remark after Thm. 2] for an explication) an isomorphism

$$\mathcal{L} \xrightarrow{\sim} \otimes_{i=0}^n \det(R\pi_*\Omega^i_{X/S})^{(-1)^i}. \tag{5.2}$$

Similarly, we may use the canonical truncations to get a distinguished triangle

$$\to R\pi_*\mathcal{H}^i(\Omega^\bullet_{X/S})[-i] \to R\pi_*\tau_{\geq i}\Omega^\bullet_{X/S} \to R\pi_*\tau_{\geq i+1}\Omega^\bullet_{X/S} \to .$$

Recall (cf., [14, §2.1]) the Cartier isomorphism $\mathcal{H}^i(F_{X/S*}\Omega^\bullet_{X/S}) = \Omega^i_{X^{(p)}/S}$, where $F_{X/S} : X \to X^{(p)}$ is the relative Frobenius map fitting in the commutative diagram with Cartesian square and $F_X = WF_{X/S}$

$$\begin{tikzcd}
X \arrow[r, "F_{X/S}"] \arrow[dr, "\pi"'] & X^{(p)} \arrow[r, "W"] \arrow[d, "\pi^{(p)}"] & X \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

where we will abuse notation and write F_S for W. Applying $R\pi^{(p)}_*$, with $\pi^{(p)} : X^{(p)} \to S$ the structure map, we get $R\pi_*\mathcal{H}^i(\Omega^\bullet_{X/S}) = R\pi^{(p)}_*\Omega^i_{X^{(p)}/S} = R\pi^{(p)}_* F_S^*\Omega^i_{X/S}$. Finally, using the base change formula $R\pi^{(p)}_* LF_S^* = F_S^* R\pi_*$ we get an identification of derived functors

$$LF_S^* R\pi_*\Omega^i_{X/S} \xrightarrow{\sim} R\pi_*\mathcal{H}^i(\Omega^\bullet_{X/S}).$$

Combining these formulas and taking determinants, we obtain an isomorphism (of graded line bundles)

$$\mathcal{L} \xrightarrow{\sim} \otimes_{i=n}^0 \det(LF_S^* R\pi_*\Omega^i_{X/S})^{(-1)^i} = F_S^*\left(\otimes_{i=n}^0 \det(R\pi_*\Omega^i_{X/S})^{(-1)^i}\right) \tag{5.3}$$

where we note the inverse order due to the Cartier isomorphism. We may then permute the last tensor product to get an isomorphism

$$F_S^*\left(\otimes_{i=n}^0 \det(R\pi_*\Omega^i_{X/S})^{(-1)^i}\right) \xrightarrow{\sim} F_S^*\left(\otimes_{i=0}^n \det(R\pi_*\Omega^i_{X/S})^{(-1)^i}\right).$$

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[1] The reader who feels the need to recall the definition of naive and canonical truncations could profitably consult [13].

Comparing the obtained formulas for $\mathcal{L}$ and $F_S^*\mathcal{L}$, we get an isomorphism $\varphi : \mathcal{L} \xrightarrow{\sim} F_S^*\mathcal{L}$; sometimes, this is called an *F-structure* on $\mathcal{L}$. Now, the isomorphism of (5.2) is compatible with duality (if we use the tensor product of the Serre duality isomorphisms on the right-hand side) and so is (5.3) because the Cartier isomorphism is multiplicative. This implies that φ is compatible with the pairing on $\mathcal{L}$. We may now consider the sheaf L (in the étale topology on S) of fixed points under φ, and we know that $\mathcal{L} = L \otimes_{\mathbf{F}_p} \mathcal{O}_S$, so that in particular, L is a local system of 1-dimensional $\mathbf{F}_p$-vector spaces. The pairing on $\mathcal{L}$ induces a symmetric non-degenerate pairing on L and taking (locally) its discriminant gives us a locally constant function from S to $\mathbf{F}_p^*/\mathbf{F}_p^{*2}$. The latter group can be identified, using the Legendre symbol, $\left(\frac{-}{p}\right)$, with $\{\pm 1\}$, and we shall call the resulting function $S \to \{\pm 1\}$ the *Hodge discriminant* of $X \to S$. It clearly commutes with base change. In particular, its value can be computed fibrewise.

The Hodge discriminant uses the whole cohomology of X/S. Often, we can also work with the middle cohomology only. Indeed, if we have an orthogonal or symplectic F-zip $(M, C^\bullet, D_\bullet, \varphi_\bullet)$, we can make the same construction, by making use of the two filtrations to identify the determinant of the middle cohomology in two ways and compare them by φ: We identify $\det M$ on the one hand with $\det \operatorname{gr}^\bullet C^\bullet$ and on the other with $\det \operatorname{gr}_\bullet D_\bullet$, then use φ to identify $F^*(\det \operatorname{gr}^\bullet C^\bullet)$ with $\det \operatorname{gr}_\bullet D_\bullet$, and finally use the induced pairing to define a discriminant for the fixed points. This yields a Hodge discriminant for the F-zip $(M, C^\bullet, D_\bullet, \varphi_\bullet)$.

Recall now (cf., [23, Def. 3.1]) the definition of Ogus' crystalline discriminant: Given an orthogonal or symplectic F-crystal M over an algebraically field $\mathbf{k}$, we get an induced F-crystal structure on $\det M$ for which F is a power of p, p^m say, times an isomorphism. Dividing by p^m and taking fixed points, we get a $\mathbf{Z}_p$-module of rank 1 with a perfect pairing. Taking its discriminant and reducing modulo p gives us an element of $\mathbf{F}_p^*/\mathbf{F}_p^{*2}$, the *crystalline discriminant.*

Before stating how these discriminants are related, we recall that for a proper smooth variety of pure dimension n over a field $\mathbf{k}$ of positive characteristic p we have the ℓ-adic Betti number $b_n(X)$ (which is the same as the rank of the n'th crystalline cohomology group), the number $b'_n(X) := \dim H^n_{dR}(X/\mathbf{k})$ and the Hodge numbers h^{ij} satisfying

$$b_n(X) \leq b'_n(X) \leq \sum_{i+j=n} h^{ij}(X).$$

If $b'_n(X) = \sum_{i+j=n} h^{ij}(X)$, then the $E_2^{i,j}$-term of the Hodge-to-de Rham spectral sequence equals the $E_\infty^{i,j}$-term for all $i + j = n$. By dimension counting this then implies the same thing for the $E_1^{i,j}$-term of the conjugate spectral sequence. Hence, the Hodge and conjugate filtrations on $H^n_{dR}(X/\mathbf{k})$ together with the Cartier isomorphisms give an F-zip structure on $H^n_{dR}(X/\mathbf{k})$. Cup product induces a non-degenerate pairing. This is symplectic if n is odd and orthogonal for n even.

Proposition 5.1 *Suppose that X is a smooth and proper variety of pure dimension n over a field* $\mathbf{k}$ *of positive characteristic p.*

i) Assume that $b'_n := \dim_{\mathbf{k}} H^n_{dR}(X/\mathbf{k}) = \sum_{i+j=n} h^{ij}(X)$. *The Hodge discriminant of* X *is equal to* $\left(\frac{-1}{p}\right)^{(c_2-b'_n)/2}$ *times the Hodge discriminant of the* F*-zip* $H^n_{dR}(X/\mathbf{k})$, *where* c_2 *is the crystalline (=étale) Euler characteristic of* X. *If* n *is odd the Hodge discriminant is equal to* $(-1)^{c_2/2}$.

ii) If $b_n(X) = \sum_{i+j=n} h^{ij}(X)$, *then the Hodge discriminant of the* F*-zip* $H^n_{dR}(X/\mathbf{k})$ *is equal to Legendre symbol of the crystalline discriminant of the F-crystal* $H^n_{cris}(X/\mathbf{W})$.

Proof We start with the easily proven fact that if $\to X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to X^{\cdot}[1] \to$ is a distinguished triangle of complexes (over a field) such the induced map $H^n(Z^{\cdot}) \to H^{n+1}(X^{\cdot})$ is zero, then we get a diagram, all of whose rows and columns are distinguished,

$$\begin{tikzcd}
{} \arrow[r] & \tau_{\le n}X^{\cdot} \arrow[r] \arrow[d] & \tau_{\le n}Y^{\cdot} \arrow[r] \arrow[d] & \tau_{\le n}Z^{\cdot} \arrow[r] \arrow[d] & (\tau_{\le n}X^{\cdot})[1] \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & X^{\cdot} \arrow[r] \arrow[d] & Y^{\cdot} \arrow[r] \arrow[d] & Z^{\cdot} \arrow[r] \arrow[d] & X^{\cdot}[1] \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \tau_{>n}X^{\cdot} \arrow[r] & \tau_{>n}Y^{\cdot} \arrow[r] & \tau_{>n}Z^{\cdot} \arrow[r] & (\tau_{>n}X^{\cdot})[1] \arrow[r] & {}
\end{tikzcd}$$

Note furthermore that it is one of the properties of the Knudsen–Mumford determinant (cf., [15, Def 4]) that the two ways of using this diagram to get an isomorphism

$$\det(Y^{\cdot}) \Longrightarrow \det(\tau_{\le n}X^{\cdot}) \otimes \det(\tau_{>n}X^{\cdot}) \otimes \det(\tau_{\le n}Z^{\cdot}) \otimes \det(\tau_{>n}Z^{\cdot}) \tag{5.4}$$

give the same result.

The degeneration of the Hodge to de Rham spectral sequence and that of the conjugate spectral sequence both at total degree $i + j = n$ implies that the necessary conditions are fulfilled to apply this and thus allow us to conclude that we have distinguished triangles:

$$\begin{aligned}
&\to \tau_{<n}R\Gamma(X, \Omega^{\ge i+1}) \to \tau_{<n}R\Gamma(X, \Omega^{\ge i}) \to \tau_{<n}R\Gamma(X, \Omega^i[-i]) \to \\
&\to \tau_{>n}R\Gamma(X, \Omega^{\ge i+1}) \to \tau_{>n}R\Gamma(X, \Omega^{\ge i}) \to \tau_{>n}R\Gamma(X, \Omega^i[-i]) \to \\
&\to \tau_{<n}R\Gamma(X, \mathcal{H}^i[-i]) \to \tau_{<n}R\Gamma(X, \tau_{\ge i}\Omega^{\bullet}) \to \tau_{<n}R\Gamma(X, \tau_{\ge i+1}\Omega^{\bullet}) \to \\
&\to \tau_{>n}R\Gamma(X, \mathcal{H}^i[-i]) \to \tau_{>n}R\Gamma(X, \tau_{\ge i}\Omega^{\bullet}) \to \tau_{>n}R\Gamma(X, \tau_{\ge i+1}\Omega^{\bullet}) \to
\end{aligned}$$

This gives us expansions

$$\begin{aligned}
\det(\tau_{<n}R\Gamma(X, \Omega^{\bullet})) &= \otimes_{i=0}^n \det(\tau_{<n-i}R\Gamma(X, \Omega^i))^{(-1)^i} \\
\det(\tau_{>n}R\Gamma(X, \Omega^{\bullet})) &= \otimes_{i=0}^n \det(\tau_{>n-i}R\Gamma(X, \Omega^i))^{(-1)^i} \\
\det(H^n(X, \Omega^{\bullet})) &= \otimes_{i=0}^n \det(H^{n-i}(X, \Omega^i))^{(-1)^i}
\end{aligned}$$

and

$$\begin{aligned}\det(\tau_{<n}R\Gamma(X,\Omega^\bullet)) &= \otimes_{i=n}^{0}\det(\tau_{<n-i}F^*R\Gamma(X,\Omega^i))^{(-1)^i}\\ \det(\tau_{>n}R\Gamma(X,\Omega^\bullet)) &= \otimes_{i=n}^{0}\det(\tau_{>n-i}F^*R\Gamma(X,\Omega^i))^{(-1)^i}\\ \det(H^n(X,\Omega^\bullet)) &= \otimes_{i=n}^{0}\det(F^*H^{n-i}(X,\Omega^i))^{(-1)^i},\end{aligned}$$

where $F = F_{\mathrm{Spec}(\mathbf{k})}$.

Now, we also have an expansion

$$\det(R\Gamma(X,\Omega^\bullet))=\det(\tau_{<n}R\Gamma(X,\Omega^\bullet))\otimes\det(H^n(X,\Omega^\bullet))\otimes\det(\tau_{>n}R\Gamma(X,\Omega^\bullet)). \tag{5.5}$$

We have already given the left-hand side an F-structure and the isomorphisms above give F-structures on each factor of the right-hand side. However, it follows from (5.4), and the fact that the tensor product on graded line bundles is symmetric monoidal (see [15, Ch. I]) that those two F-structures are the same.

As for the self-pairing, the duality induces isomorphisms $(\tau_{<n}R\Gamma(X,\Omega^\bullet_X))^* = \tau_{>n}R\Gamma(X,\Omega^\bullet_X)[-2n]$ and $H^n(X,\Omega^\bullet_X)^* = H^n(X,\Omega^\bullet_X)$. This means that in the decomposition of (5.5), the self-pairing on the left corresponds to a pairing on the right which pairs the first factor to the third and the second to itself. This is compatible with semi-linear structure so that when we take fixed points under F, we get a decomposition $L = L^< \otimes L^= \otimes L^>$, with $L^<$ and $L^>$ being paired to each other and $L^=$ to itself. Taking the signs into account, we get that the discriminant of L is equal to $\left(\frac{-1}{p}\right)^{\dim L^<}$ times the discriminant of $L^=$. However, the dimension of $L^<$ is equal to the dimension of $\tau_{>n}R\Gamma(X,\Omega^\bullet_X)$ which is $(c_2 - b'_n)/2$. Of course, when n is odd $L^=$ is trivial. This concludes the proof of i).

As for ii) we are reduced by i) to showing that the Hodge discriminant of $H^n_{dR}(X/\mathbf{k})$ is equal to the crystalline discriminant of $H^n_{cris}(X/\mathbf{W})$. By the Mazur–Ogus result [5, Appendix], we may choose a basis $e^1_1,\dots,e^1_{h^1},e^2_1,\dots,e^r_{h^r}$ of $H^n_{cris}(X/\mathbf{W})$ such that Fe^s_j is divisible by p^s and such that the Hodge filtration of $H^n_{dR}(X/\mathbf{k})$ is given by $H_i = \sum_{j\le i}\sum_{1\le k\le h^j}\mathbf{k}\bar{e}^j_k$, the conjugate filtration is given by $H^c_i = \sum_{j\le i}\sum_{1\le k\le h^j}\mathbf{k}\overline{p^{-j}Fe^j_k}$ and the inverse Cartier isomorphism is given by $\{p^{-j}F\}$. Unraveling definitions then gives part ii). □

Remark 5.2 As the proposition shows, under suitable conditions the Hodge discriminant is equivalent to the crystalline discriminant. We justify introducing new notation since it would be somewhat artificial to use "crystalline" in a situation where it is not relevant; moreover, it makes sense more generally, for instance in the case of Enriques surfaces in characteristic two, cf. [11]. Please note that we have defined the Hodge discriminant to be the Legendre symbol applied to an element of $\mathbf{F}^*_p/\mathbf{F}^{*2}_p$ rather than the element itself. (Of course the Legendre symbol gives a bijection with this group and $\{\pm 1\}$ so no information is lost.) The reason for this convention is to make the formulas of Proposition 5.3 as nice as possible; otherwise, that formula would have to involve the inverse of the isomorphism provided by the Legendre symbol.

The further properties of the Hodge discriminant will differ somewhat depending on whether H has even or odd dimension, so we shall discuss each case separately.

5.1 The Hodge discriminant of an orthogonal flagged F-zip

We now derive a formula for the Hodge discriminant of an orthogonal flagged F-zip. In the odd-dimensional case, we need one more notion. Hence, consider a flagged orthogonal F-zip $(H, C^\bullet, D_\bullet, \varphi_\bullet)$ of dimension $2m+1$. We get two induced isomorphisms

$$C^{m-1}/C^m \xleftarrow{\sim} C^{m-1} \cap D_{m+1}/C^m \cap D_m \xrightarrow{\sim} D_{m+1}/D_m$$

and together with the inverse Cartier isomorphism they give rise to an isomorphism

$$F^*(C^{m-1}/C^m) \xrightarrow{\sim} C^{m-1}/C^m.$$

On the other hand, we also have a pairing on C^{m-1}/C^m induced from that of H and it is compatible with $F^*(C^{m-1}/C^m) \xrightarrow{\sim} C^{m-1}/C^m$. Hence, we get an $\mathbf{F}_p^*/\mathbf{F}_p^{2*}$-valued discriminant by taking fixed points. We shall call it the *middle discriminant.*

Proposition 5.3 *Let $(H, C^\bullet, D_\bullet, \varphi_\bullet)$ be an orthogonal flagged F-zip.*

i) Assume H has dimension $2m+1$ of type $w \in W_m^C$. Then the Hodge discriminant equals $(-1)^{n_s}\left(\frac{d}{p}\right)\mathrm{disc}(w)$, where $d = (-1)^m d'$ with d' the middle discriminant, and $s = [r/2]$.

ii) Assume H has dimension $2m$ of type $w \in W_m'^D$. Then the Hodge discriminant equals $(-1)^{n_s}\left(\frac{-1}{p}\right)^m\mathrm{disc}(w)$, where $s = [r/2]$.

Proof For the odd case we may assume by Proposition 4.1 that the F-zip is associated to a $\mathbf{k}$-point of Y_w, i.e., there is a basis $e_1, \ldots, e_{2m+1}$ for H with $\langle e_i, e_j\rangle = \delta_{i,2m+2-j}$, $F_i = \sum_{j\le i} \mathbf{k}e_j$, and $D_i = \sum_{j \le i} \mathbf{k}e_{w^{-1}(j)}$ with φ_k acting as specified by the construction of the F-zip on Y_w. This implies that $\varphi_k(e_{n_k+1} \wedge e_{n_k+2} \wedge \cdots \wedge e_{n_{k+1}}) = \epsilon_k e_{w^{-1}w_\emptyset(n_k+1)} \wedge e_{w^{-1}w_\emptyset(n_k+2)} \wedge \cdots \wedge e_{w^{-1}w_\emptyset(n_{k+1})}$. Here $\epsilon_k = 1$ when $k \ne m$ and $\epsilon_m = \pm 1$ with $+1$ when the middle discriminant is a square and -1 when it is not. This implies that the semi-linear map on the determinant takes $e_1 \wedge e_2 \wedge \cdots \wedge e_{2m+1}$ to $\epsilon_m e_{w^{-1}w_\emptyset(1)} \wedge e_{w^{-1}w_\emptyset(2)} \wedge \cdots \wedge e_{w^{-1}w_\emptyset(2m+1)} = \epsilon_m \mathrm{disc}(w^{-1}w_\emptyset) e_1 \wedge e_2 \wedge \cdots \wedge e_{2m+1}$. Similarly, we have that $\langle e_1 \wedge e_2 \wedge \cdots \wedge e_{2m+1}, e_1 \wedge e_2 \wedge \cdots \wedge e_{2m+1}\rangle = (-1)^m$. Now, we conclude by the mod p version of [23, Formula 3.4] using that d' is a square precisely when is ϵ_m and that $\mathrm{disc}(w_\emptyset) = (-1)^{n_s}$.

The proof of the even-dimensional case is identical to the odd case except that we always have $\epsilon_m = 1$. □

6 K3 surfaces

In this section, we shall consider the primitive cohomology of a polarized K3 surface and show that it possesses a minimal *stable* filtration that refines the Hodge filtration and that it can be refined to a so-called *final* filtration if the field of definition is separably closed.

The results of the preceding section can be applied because the crystalline cohomology of a K3 surface is without torsion and the Hodge-to-de Rham spectral sequence degenerates at the E_2-level, cf. [9,14].

Recall that for a K3 surface the Néron–Severi group $\mathrm{NS}(X)$ is equal to the Picard group of X. Let N be a non-degenerate integral lattice. A *(partial) N-marking* of a K3 surface X over a field $\mathbf{k}$ of positive characteristic p is an isometric embedding $N \to \mathrm{NS}(X)$. The *discriminant* of the marking is the discriminant of the lattice N. We shall only be interested in partial markings whose degree (order of the discriminant group) is prime to p and thus that will be assumed unless otherwise mentioned. We define the *primitive cohomology* of an N-polarized K3 surface X as the orthogonal complement of the image of N in $H^2_{dR}(X/\mathbf{k})$.

The primitive cohomology is an orthogonal F-zip with dimension vector for its Hodge filtration being $(0, 1, n-1, n)$ for some n. We shall also need another type of F-zip. Namely, an F-zip of dimension type $(0, \dots, 0, m, \dots, m)$ shall be called a *Tate F-zip*. Tate F-zips thus consist of an orthogonal vector space V and an orthogonal F-structure $\varphi : F^*V \to V$. It is thus completely described by the orthogonal representation of the Galois group of $\mathbf{k}$ given by the action on $\mathcal{V} := \ker(\varphi - 1)$ on $V \otimes_{\mathbf{k}} \overline{\mathbf{k}}$. We shall say that the Tate F-zip is *split* resp. *non-split* as the form on $\mathcal{V}$ is. Its Hodge discriminant is then $\left(\frac{d}{p}\right)$, where d is the discriminant of $\mathcal{V}$. In these terms, we have that $H^2_{dR}(X/\mathbf{k})$ is the sum, as F-zip, of the primitive cohomology and $N \otimes \mathbf{k}$ considered as a Tate F-zip.

Definition 6.1 Let M be a stable partially flagged orthogonal or symplectic F-zip.

i) M is *final* if it is complete.
ii) If M is symplectic or orthogonal of odd dimension then it is *canonical* if every stable flag is a refinement of it.
iii) If M is orthogonal of even dimension it is *canonical* if every stable flag is a refinement of M or possibly, when it exists, its flip.

Example 6.2 For a Tate F-zip its (trivial) Hodge filtration already is canonical. A final filtration is an $\mathbb{F}_p$-rational self-dual flag except in case the $\mathbb{F}_p$-form is non-split. Then, the middle element of the flag is only defined over $\mathbb{F}_{p^2}$.

Lemma 6.3 *Let w be an element of W^B_m or W'^D_m. Assume that for all $1 \le i, j \le n-1$, where we do not have $i = j = n/2$,*

$$r_w(i, j) = \begin{cases} \min(j, r_w(i, n-1) + 1) - 1 & \text{if } i < a, \\ \min(j, r_w(i, n-1)) & \text{if } i \ge a, \end{cases}$$

where $a := w^{-1}(1)$ and $n = 2m + 1$ in the B-case and $2m$ in the D-case. Then w is a final element and conversely the r_w for w final fulfils this condition.

Proof By definition we have $r_w(i, n-1) = \#\{1 \le b \le i : w(b) \le n-1\}$. Thus, it is clear that it is determined completely by $w^{-1}(n)$ which is equal to $n + 1 - a$. The assumed conditions on r_w then implies that the whole function is determined by a and hence so is w. It is easy to verify that a final w fulfills the conditions and that there is one such element for each a.

Recall that a flagged F-zip has a type which is an element w of a Weyl group. We now link the definition of a final F-zip to the notion of a final element of a Weyl group.

Proposition 6.4 *A flagged orthogonal F-zip of type $(0, 1, n-1, n)$ is final precisely when its type is a (twisted) final element.*

Proof Suppose that $E_\bullet$ is a final filtration and $G_\bullet$ the corresponding conjugate filtration (so that $0 \subset E_1 \subset E_{n-1} \subset E_n$ is the Hodge filtration and $0 \subset G_1 \subset G_{n-1} \subset G_n$ the conjugate filtration). For each $1 \le i \le n-1$ we have by assumption that for every i the subspace $G_i \cap E_{n-1} + E_1$ is equal to some E_r (or possibly its flip E'_r if $2r = n$). Then for $1 \le j \le n-1$

$$G_i \cap E_j + E_1 = (G_i \cap E_{n-1} + E_1) \cap E_j = E_r \cap E_j = E_{\min(r,j)},$$

where the end result would instead be E_{r-1} if $G_i \cap E_{n-1} + E_1 = E'_r$ and $j = r$. Now, $r_w(i, j) = \dim(G_i \cap E_j)$ and in particular $E_1 \subseteq G_i$ precisely when $i \ge w^{-1}(1)$ and thus $\dim(G_i \cap E_j + E_1)$ is equal to $r_w(i, j) + 1$ if $i < w^{-1}(1)$ and $r_w(i, j)$ otherwise (supposing that we do not have $i = j = n/2$). This shows that r_w fulfills the condition of Lemma 6.3 and hence w is final. The converse is just a matter of tracing the argument backwards. □

Recall that two orthogonal F-zips of type $(0, 1, n-1, n)$ are called *opposite* if their intersections have the smallest possible dimensions, i.e., $F_1 \not\subset E_{n-1}$. It follows from either the description of the final elements or from the proof of the next theorem that the canonical filtrations have the form $U_1 \subset U_2 \subset \cdots \subset U_k \subset U_{n-k} \subset \cdots \subset U_n$, where the primitive cohomology has dimension n. We shall call U_{n-k}/U_k the *middle part* of the canonical filtration. It comes equipped with a quadratic form induced from that of $H^2_{dR}(X/\mathbf{k})$ and the Cartier isomorphism induces an orthogonal p-linear isomorphism of it (i.e., a Tate F-zip structure). The fixed points under the Cartier isomorphism then give an $\mathbb{F}_p$-rational structure on the middle part, and the quadratic form induces a quadratic form on it. We shall say that the canonical filtration is *split* resp. *non-split* according to as that form is.

Theorem 6.5 *Let X be a polarised K3 surface of degree prime to p over a field $\mathbf{k}$ of characteristic $p > 0$ and let H be its primitive Hodge cohomology of dimension n with $m := [n/2]$. Then H has a canonical filtration. Any final filtration is obtained from the canonical one by choosing a complete F-stable filtration. If $\mathbf{k}$ is separably closed H has a final filtration. All final filtrations have the same (twisted) final type.*

Proof We start with the induced Hodge filtration $0 \subset E_1 \subset E_{n-1} \subset E_n = H$ on the primitive cohomology with conjugate filtration $0 \subset F_1 \subset F_{n-1} \subset F_n = H$ with $F_i = E^c_{n-i}$. If $F_1 = E_1$, then the two filtrations coincide and then this partially flagged F-zip is canonical as one easily checks. If $F_1 \ne E_1$, then we consider the image of F_1 in E_n/E_{n-1}. If this image is nonzero, then the Hodge filtration and the conjugate one are opposite and we get a stable and hence canonical flagged F-zip. So, suppose that F_1 has non-zero image in E_{n-1}/E_1. We can apply Frobenius and use the Cartier isomorphism to get the image $\overline{F}_2$ in $E^c_{n-1}/E^c_1 = F_{n-1}/F_1$. We then add to our flag the inverse image F_2 of $\overline{F}_2$ in F_{n-1}. Now, F_1 is totally isotropic; hence, its image in E_{n-1}/E_1 is as well and as the Cartier isomorphism is multiplicative for the wedge product, so is F_2 and therefore $F_{n-2} := F_2^{\perp}$ contains F_2. We then continue this process: If F_2 is not contained in E_{n-1}, then we have obtained a stable filtration. This is also the case if $E_1 \subset F_2$. On the other hand, if $F_2 \subset E_{n-1}$ and $F_2 \cap E_1 = \{0\}$, we consider the image of F_2 in E_{n-1}/E_1 and transfer via F and the Cartier isomorphism the image to F_{n-1}/F_1. Note that each stage of the induction $F_{n-i} \subset E_{n-1}$ precisely when $E_1 \subset F_i$, and thus, we will not be forced to introduce any new elements to the flag because of the position of $F_{n-i} := F_i^{\perp}$. It is clear that this process stops and gives a canonical flag. That a canonical filtration can be extended to a final filtration and that they are all of the same type is easy and similar to the abelian case, see [10, Def-Lemma 2.11]. ◻

7 Canonical filtrations versus the height and Artin invariant

As we just saw, the primitive part of the 2nd de Rham cohomology of a K3 surface in positive characteristic comes with a canonical filtration. If our field is separably closed, we can refine it to a final filtration. A natural question is now what the type of the final filtration means geometrically. The following theorem will provide the answer. It relates the relative position of a final filtration and its conjugate one, given by an element w of a Weyl group, to geometric invariants. Recall that we have two invariants for a K3 surface X in positive characteristic, the *height* and if the height is infinite we also have the so-called *Artin invariant*. The height $h(X)$ is the height of the formal Brauer group, a smooth formal group of dimension 1. This invariant assumes values $1 \le h \le 10$ or $h = \infty$; in the latter case, the formal Brauer group is the formal additive group. The Artin invariant σ_0 can be defined for supersingular K3 surfaces, i.e., those with $h = \infty$ by putting $\mathrm{disc}(H^2(X, \mathbb{Z}_p(1))) = -p^{2\sigma_0}$, cf., [2]. We then have $1 \le \sigma_0 \le 10$. The case $h = 1$ is called the ordinary case, and it is the generic finite height case and $\sigma_0 = 10$ is the generic supersingular case.

These invariants can be detected by the crystalline cohomology. We therefore recall first some facts on crystalline cohomology and the relation with de Rham cohomology (cf., [5, Thm 8.26]).

Let $\mathbf{W}(\mathbf{k})$ be the ring of Witt vectors of $\mathbf{k}$, and let σ be the map on $\mathbf{W}(\mathbf{k})$ induced by the Frobenius map on $\mathbf{k}$. The second crystalline cohomology group $\mathcal{H} := H^2(X/\mathbf{W}(\mathbf{k}))$ is a free $\mathbf{W}(\mathbf{k})$-module of rank 22 and is provided with a $\mathbf{W}(\mathbf{k})$-linear map $F : \sigma^*\mathcal{H} \to \mathcal{H}$. We have a natural isomorphism from $\mathcal{H}/p\mathcal{H} \xrightarrow{\sim} H^2_{dR}(X/\mathbf{k})$ and by base change by the Frobenius map on $\mathbf{k}$ we get an isomorphism $\sigma^*\mathcal{H}/p\sigma^*\mathcal{H} \xrightarrow{\sim} H^2_{dR}(X^{(p)}/\mathbf{k})$. If we put $\mathcal{H}_i := F^{-1}p^{2-i}\mathcal{H}$ for $i = 0, 1, 2$, then the images H_i of the $\mathcal{H}_i$ in $\sigma^*\mathcal{H}/p\sigma^*\mathcal{H}$ give the Hodge filtration on $H^2_{dR}(X^{(p)}/\mathbf{k})$ (with $E_1 = H_0$, $E_{n-1} = H_1$ and $E_n = H_2$ in the notation of the proof of Thm 6.5) while the images of the $\mathcal{H}^c_i := p^{-i}F\mathcal{H}_{2-i}$ in $H^2_{dR}(X/\mathbf{k})$ give the conjugate filtration. Finally, the inverse Cartier isomorphism is induced by the map $p^{-i}F : \mathcal{H}_{2-i} \to \mathcal{H}^c_i$.

We now give the main result connecting the final type with the classical invariants (height and Artin invariant) and the Hodge discriminant.

Recall that we choose a marking by giving an isometric embedding $N \to \mathrm{NS}(X)$. In particular, we have a discriminant d of the marking.

Theorem 7.1 *Let X be a polarized K3 surface of degree prime to p over a field $\mathbf{k}$ of characteristic $p > 0$ and let H be its primitive Hodge cohomology of dimension n with $m := [n/2]$.*

i) *If X has finite height h with $2h < n$, then H has final type w_h or w'_h. When n is even it is w_h if the middle part is non-split and w'_h if it is split.*
ii) *If X has finite height $h = n/2$, then H has final type w'_m.*
iii) *If X is supersingular with Artin invariant $\sigma_0 < n/2$, then H has final type $w_{2m+1-\sigma_0}$ or $w'_{2m+1-\sigma_0}$. When n is even it is $w_{2m+1-\sigma_0}$ if the middle part is split and $w'_{2m+1-\sigma_0}$ if it is non-split.*
iv) *If X is supersingular with Artin invariant $\sigma_0 = n/2$, then H has final type w_{m+1}.*
v) *The Hodge discriminant of H is equal to $\left(\frac{-d}{p}\right)$, where d is the discriminant of the marking.*

Proof Note that because the discriminant of the marking is prime to p, our space $\mathcal{H} := H^2(X/\mathbf{W}(\mathbf{k}))$ splits into the orthogonal direct sum $(\mathbf{W}(\mathbf{k})\otimes N^{\perp})\oplus(\mathbf{W}(\mathbf{k})\otimes N)$, where N embeds using the crystalline Chern class and a similar statement is true for Hodge cohomology. This gives in particular that the Hodge discriminant of $H^2_{dR}(X/\mathbf{k})$ is the product of the Hodge discriminant of the primitive part and the Legendre symbol of the discriminant of N/pN. By a theorem of Bloch and Ogus (cf., [24, Thm 4.9]), it is equal to $(-1)^{22-1}$ and this together with the relation between the Hodge and crystalline discriminants gives v).

Now, if we perform our construction of the canonical filtration on all of $H^2_{dR}(X/\mathbf{k})$, it will be performed separately on the reduction modulo p of $\mathbf{W}(\mathbf{k})\otimes N^{\perp}$ and $\mathbf{W}(\mathbf{k})\otimes N$. Furthermore, it will be completely trivial on the second factor having a canonical flag consisting only of the zero subspace and the full space. Hence, we may as well work with the full crystalline and de Rham cohomologies rather than their primitive parts and we shall do exactly that. Thus, now H may be identified with $\mathcal{H}/p\mathcal{H}$. With these results in mind, we shall now consider the different cases.

Case 1 Consider first the case of finite height h. Then $\mathcal{H}$ splits as an orthogonal F-stable direct sum

$$M_{1/h} \oplus \mathbf{W}(\mathbf{k})^{22-2h}(1) \oplus M_{2-1/h}.$$

Here, $M_{1/h}$ is the crystalline Dieudonné module with basis $e_1, e_2, \ldots, e_{h-1}$ where $Fe_i = p\,e_{i-1}$ for $i = 2, \ldots, h-1$ and $Fe_1 = e_{h-1}$. Further, $\mathbf{W}(\mathbf{k})(1)^{22-h}$ is free of rank $22-2h$ with F acting as p on a basis. Finally, $M_{2-1/h}$ is the dual $M^*_{1/h}(1)$ of $M_{1/h}$ as Dieudonné module twisted once (i.e., the Frobenius map is multiplied by p). In particular, $M_{2-1/h}$ has a basis $f_1, f_2, \ldots, f_{h-1}$ with $Ff_i = pf_{i+1}$ for $i = 1, \ldots, h-2$ and $Ff_{h-1} = p^2 f_1$. Furthermore, we have an orthogonal decomposition $M_{1/h} \oplus M_{2-1/h} \perp \mathbf{W}(\mathbf{k})^{22-2h}(1)$ which again means that the Hodge and conjugate filtrations will be a direct sum of those of the summands. As before the filtrations on the $\mathbf{W}(\mathbf{k})^{22-2h}(1)$ factor will be trivial and we hence may restrict to the other factor and will put $\mathcal{H}$ equal to $M_{1/h} \oplus M_{2-1/h}$. There the pairing will be given by identifying $M_{2-1/h}$ with the dual of $M_{1/h}$. From the description above, we conclude that (employing the notation $(M_{1/h})_i = M_{1/h} \cap \mathcal{H}_i$ for $i = 0, 1$ with $\mathcal{H}_i$ as defined above)

$$\begin{aligned}
(M_{1/h})_1 &= p\mathbf{W}e_1 + \mathbf{W}e_2 + \cdots + \mathbf{W}e_{h-1},\\
(M_{2-1/h})_1 &= M_{2-1/h},\\
(M_{1/h})_0 &= p^2\mathbf{W}e_1 + p\mathbf{W}e_2 + \cdots + p\mathbf{W}e_{h-1},\\
(M_{2-1/h})_0 &= p\mathbf{W}f_1 + \cdots + p\mathbf{W}f_{h-2} + \mathbf{W}f_{h-1}.
\end{aligned}$$

This implies that H_0 has $\overline{f}_{h-1}$ as a basis and H_1 has $\overline{e}_2, \ldots, \overline{e}_{h-1}, \overline{f}_1, \ldots, \overline{f}_{h-1}$ as a basis. Similarly, we get that H_0^c has $\overline{e}_{h-1}$ as a basis and H_1^c has $\overline{e}_1, \ldots, \overline{e}_{h-1}, \overline{f}_1, \ldots, \overline{f}_{h-2}$ as a basis and we also see that $C^{-1} : F^*(H_1/H_0) \to H_1^c/H_0^c$ takes $\overline{e}_i$ to $\overline{e}_{i-1}$ for $1 \leq i \leq h-1$ and $\overline{f}_i$ to $\overline{f}_{i+1}$ for $0 \leq i \leq h-2$.

Now, as we saw during the construction of the canonical filtration, we do not need to introduce $U_{n-i} := U_i^{\perp}$ of our desired filtration at each stage of the construction but can do it when the construction is finished. From the description above, it follows that $H_0 + H_0^c = \mathbf{k}\overline{e}_{h-1} + \mathbf{k}\overline{f}_{h-1}$. Transferring by the Cartier isomorphism forces us to add $\mathbf{k}\overline{e}_{h-2} + \mathbf{k}\overline{e}_{h-1}$ to the refinement of the conjugate filtration. Continuing one sees that all the $\mathbf{k}\overline{e}_{h-i} + \cdots + \mathbf{k}\overline{e}_{h-1}$ for $i \leq h$ must be added to the canonical filtration. Then also their annihilators $\mathbf{k}\overline{f}_{n-i} + \cdots + \mathbf{k}\overline{f}_{h-1} + \overline{M_{1/h}}$ must be added. We thus get a canonical filtration with the property that $\mathbf{k}^{22-2h} = \overline{\mathbf{W}(\mathbf{k})^{22-2h}(1)}$ maps isomorphically to the quotient of the h'th and $h+1$'st step in the filtration. We can complete such a canonical flag by adding a complete self-dual flag of $\overline{\mathbf{W}(\mathbf{k})^{22-2h}(1)}$ which is fixed under $p^{-1}F$ i.e., an $\mathbb{F}_p$-rational such flag. By comparing our zip to the standard case of Proposition 3.1 and the form of the final elements one sees directly that these are of type w_h or w'_h. To decide whether the form on $\mathbb{F}_p^{22-2h}$ is split or not, we interpret its discriminant in terms of the crystalline discriminant (cf., [23]), i.e., the discriminant of the fixed points of $p^{2h-22}F$ on $\Lambda^{22-2h}(\mathbf{W}(\mathbf{k})^{22-2h}(1))$ multiplied by $(-1)^{11-h}$. As $\mathcal{H}$ splits up as the orthogonal direct sum of $\mathbf{W}(\mathbf{k})^{22-2h}(1)$ and a hyperbolic space on $M_{1/h}$, we see that the crystalline discriminant of $\mathcal{H}$ equals the product of $(-1)^h$ and the crystalline discriminant of $\mathbf{W}(\mathbf{k})^{22-2h}(1)$. It follows from Proposition 5.3 that if the type is w, then the Hodge discriminant is $-\left(\frac{-1}{p}\right)^{11}\mathrm{disc}(w)$. Now, from the Bloch–Ogus theorem and what we just proved, we get that the Hodge discriminant of the middle part is $-\left(\frac{-1}{p}\right)^{h}\mathrm{disc}(w)$ and as it is split precisely when its Hodge discriminant is $\left(\frac{-1}{p}\right)^{11-h}$ we see that it is split precisely when $\mathrm{disc}(w) = -1$.

Case 2 Turning now to the case of infinite height let us recall the setup of [22]. (As we do not want to assume that $\rho = 22$ we shall, however, replace $\mathrm{NS} \otimes \mathbb{Z}_p$ by the flat cohomology group $H^2(X, \mathbb{Z}_p(1))$, which it is equal to when $\rho = 22$)[2]. We let

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[2] Now that Artin's conjecture has been proved for $p \geq 5$ one might work as well with $\mathrm{NS} \otimes \mathbb{Z}_p$.

N be the flat cohomology group $H^2(X, \mathbb{Z}_p(1))$ and consider $N \otimes \mathbf{k}$ with F acting as id $\otimes F$. We then have de Rham Chern class map $c_1 : N \to H^2_{dR}(X/\mathbf{k})$, and we shall also write c_1 for the $\mathbf{k}$-linear extension $N \otimes \mathbf{k} \to H^2_{dR}(X/\mathbf{k})$ of c_1. The kernel of this map is called characteristic subspace and plays the central role, see [22]. We write the kernel of it on the form F^*K for some sub-vector space $K \subseteq N \otimes \mathbf{k}$. We let $\tilde{K}$ be the inverse image of K in $N \otimes \mathbf{W}$. We can consider $\mathcal{H}$ as a $\mathbf{W}$-submodule of $N \otimes Q$, where Q is the fraction field of $\mathbf{W}$. Then, by definition and the fact that $pN^* \subseteq N$, with N^* the dual of N with respect to the intersection pairing, we have that $\mathcal{H} = p^{-1}\sigma^*\tilde{K}$. Furthermore, as $F = p \otimes \sigma$ on $N \otimes \mathbf{W}$ it is clear that

$$\mathcal{H}_1 = F^{-1}p\mathcal{H} = p^{-1}(\tilde{K} \cap \sigma^*\tilde{K}) \quad \text{and} \quad \mathcal{H}_0 = F^{-1}p^2K = \tilde{K}$$

and they map to the Hodge filtration of H. On the other hand,

$$\mathcal{H}_0^c = F(\mathcal{H}_2) = \sigma^{*2}(\tilde{K}) \quad \text{and} \quad \mathcal{H}_1^c = p^{-1}F(\mathcal{H}_1) = p^{-1}(\sigma^*\tilde{K} \cap \sigma^{*2}\tilde{K})$$

which then map to the conjugate filtration. Starting our procedure for constructing the canonical filtration, we see that it stops immediately when $H_0 = H_0^c$, but this is the case precisely when the Artin invariant σ_0 equals 1. If not, we add the image $\overline{E}_2$ of H_0^c in H/H_0 to the Hodge filtration, whose inverse image in $\mathcal{H}$ then is $\tilde{U}_2 := \tilde{K} + \sigma^*\tilde{K} + \sigma^{*2}\tilde{K}$. The next step is to transfer $\overline{E}_2$ via the Cartier isomorphism to get an addition, V_2, to the conjugate filtration. As the Cartier isomorphism between the "middle parts" of the Hodge and conjugate filtrations is implemented by $p^{-1}F = \sigma^*$, we get that the inverse image of V_2 in H is given by

$$\sigma^*\tilde{K} + \sigma^{*2}\tilde{K} + \sigma^{*3}\tilde{K}.$$

The process stops at that stage precisely when $\sigma^{*3}\tilde{K} \subseteq \tilde{K} + \sigma^*\tilde{K} + \sigma^{*2}\tilde{K}$ which in turn is equivalent to $\tilde{K} + \sigma^*\tilde{K} + \sigma^{*2}\tilde{K}$ being stable under σ^*. However, that in turn is equivalent to $\tilde{K} + \sigma^*\tilde{K} + \sigma^{*2}\tilde{K} = pN^* \otimes \mathbf{W}(\mathbf{k})$ (by [22, 3.12.3]) and thus to $\sigma_0 = 2$ (as we have put ourselves in the case when $\sigma_0 > 1$). If not, the process continues, forcing us to add the image of $\sigma^*\tilde{K} + \sigma^{*2}\tilde{K} + \sigma^{*3}\tilde{K}$ to the Hodge filtration. If we continue in this way, it is clear that we will stop at $\tilde{K} + \sigma^*\tilde{K} + \cdots + \sigma^{*\sigma_0}\tilde{K}$ which equals $pN^* \otimes \mathbf{W}(\mathbf{k})$. In this way, we get an extension of the Hodge filtration which ends at $R \otimes \mathbf{k}$, where R is the radical of $\overline{N} := N \otimes \mathbb{F}_p$. Its annihilator is $N \otimes \mathbf{k}$, and hence, we get that the "middle subquotient" of the canonical filtration is canonically isomorphic to $\overline{N}/R \otimes \mathbf{k}$ with the natural quadratic structure and the map induced by the Cartier isomorphism having $\overline{N}/R$ as its fixed points. Hence, extending the canonical filtration to a final one amounts to finding a complete self-dual flag in $\overline{N}/R$. We also get from [22, 3.4] and the fact that the discriminant of $N(X)$ is $-p^{2\sigma_0}$ that the quadratic form on $\overline{N}/R$ is non-split. Also, in this case, it is evident by inspection from the filtration thus obtained that it is of type $w_{n-1-\sigma_0}$ or $w'_{n-1-\sigma_0}$. In the case where $\sigma_0 = n/2$ we find $w_{n-1-\sigma_0}$. In the general case, we see that the F-zip is the sum of an F-zip of dimension $2\sigma_0$ and a Tate F-zip which is isomorphic to the middle part F-zip. From the multiplicativity of the Hodge discriminant and the case $n = 2\sigma_0$, we conclude.

□

8 Strata on the flag space

Here, we shall define strata on the flag space of orthogonal or symplectic flags on the primitive part of the second de Rham cohomology of a family of polarized K3 surfaces. Hence, we assume that we have a family $f: X \to S$ of N-marked K3 surfaces (where S may be an algebraic stack). The primitive cohomology forms a vector bundle $\mathcal{H}$ over S of rank n with an orthogonal structure given by the intersection form. It is provided with two orthogonal partial flags: the Hodge flag and the conjugate flag, thus giving a F-zip.

If we choose an orthogonal flag refining the Hodge filtration $C^\bullet$, we obtain by using the Cartier isomorphism $\varphi_i : F^*(C^i/C^{i+1}) \cong D_{i+1}/D_i$ a second flag refining the conjugate filtration $D_\bullet$.

We let $\mathcal{F}_n$ be the space of complete orthogonal flags on $\mathcal{H}$ refining the Hodge filtration. It admits a canonical projection $\mathcal{F}_n \to S$. Since a flag refining the Hodge filtration automatically defines a second flag (refining the conjugate filtration), we can measure the relative position of these flags and thus define strata on S. We refer to [12] for background.

We can formulate this in the following way, cf. [10]. Let G be a semi-simple group and B a Borel subgroup and G/B-bundle $T \to Y$ over some scheme Y with G as structure group. Suppose that we have two sections $t_i : Y \to T$ of T with $i = 1, 2$. If w is an element of the Weyl group of G, we define a locally closed subscheme $\mathcal{U}_w$ (resp. $\overline{\mathcal{U}}_w$) of Y by choosing locally (possibly in the étale topology) a trivialization of T for which t_1 is a constant section. Then, t_2 corresponds to a map $Y \to G/B$, and we define $\mathcal{U}_w$ (resp. $\overline{\mathcal{U}}_w$) to be the inverse image of the B-orbit BwB (resp. of its closure in G/B). This does not depend on the trivialization taken since the difference corresponds to a map $Y \to B$, and the cycles BwB and its closure are B-invariant. Therefore, this defines global subschemes $\mathcal{U}_w$ (resp. $\overline{\mathcal{U}}_w$) of Y. If t_1 and t_2 have the property that $Y = \overline{\mathcal{U}}_w$, then we say that t_1 and t_2 are in relative position w. We thus find our strata $\mathcal{U}_w$ and $\overline{\mathcal{U}}_w$ associated with an element of our Weyl group. Note that a priori it is not clear that the closure of $\mathcal{U}_w$ equals $\overline{\mathcal{U}}_w$, but this will hold (see below).

We shall apply this to the situation that $\mathcal{F}_n$ is the space of orthogonal flags refining the conjugate filtration on $\mathcal{H}$ for the family $f : X \to S$.

Definition 8.1 On the space $\mathcal{F}_n$ of orthogonal flags on $\mathcal{H}$ we define for every element w in our Weyl group the locally closed subschemes $\mathcal{U}_w$ and $\overline{\mathcal{U}}_w$ of $\mathcal{F}_n$ associated to the flag refining the conjugate filtration and the induced flag on the Hodge filtration as the subschemes that measure the relative position of these two orthogonal flags as defined above. For final elements w in our Weyl group we have an orthogonal flag refining the canonical filtration and then define the stratum $\overline{\mathcal{V}}_w$ on S as the locally closed subset of S of points for which the canonical type of the K3 surface is equal to the canonical type of w. By Theorem 7.1 these strata $\overline{\mathcal{V}}_w$ belong to the height and Artin invariant.

It might seem that working on the flag space $\mathcal{F}_n$ rather than on S is a detour, but in the next section, we shall see that it helps understanding the strata on the moduli.

9 The local structure of strata

The reason for working on the flag space over our moduli space (of lattice polarized K3 surfaces) is that the strata are much better behaved than on the moduli space. In fact, up to infinitesimal order $<p$ the strata look like usual Schubert strata. This idea of [10] and the methods employed there can be transferred to our situation. Hence, we assume that we have a family $f: X \to S$ of N-marked K3 surfaces (where S may be an algebraic stack). We shall also need to assume a versality condition: For any geometric point s of S contraction of forms by vector fields induces a map $H^1(X_s, T_{X_s}) \to \mathrm{Hom}(H^0(X_s, \Omega^2_{X_s}), H^1(X_s, \Omega^1_{X_s}))$ we can then compose this with the map induced by the projection on the second factor of the decomposition $H^1(X_s, \Omega^1_{X_s}) = N \otimes \mathbf{k} \perp P$ and then further compose the resulting map with the Kodaira–Spencer map $T_s S \to H^1(X_s, T^1_{X_s})$. The required versality condition is that S be smooth at all s and that the composed map $T_s S \to \mathrm{Hom}(H^0(X_s, \Omega^2_{X_s}), P)$ be surjective.

The space $\mathcal{F}_n$ together with the $\mathcal{U}_w$ is a stratified space. The space $\mathcal{F}\ell_n$ of complete self-dual flags on an orthogonal space is also a stratified space with the stratification given by the Schubert cells. The idea is that our flag space at a point can be identified up to the $(p-1)$st neighborhood with the flag space at an appropriate point. Moreover, under this correspondence, the strata on $\mathcal{F}_n$ correspond precisely to the Schubert strata on $\mathcal{F}\ell_n$. This enables us to transplant the detailed knowledge about Schubert strata up to order p to our situation. More precisely, if R is a local ring with maximal ideal m defining an affine scheme S, then the height-1 hull of R (or S) is given by $R/m^{(p)}$, with $m^{(p)}$ generated by the p'th powers of elements of m. We call two local rings *height 1-isomorphic* if their respective height 1-hulls are isomorphic.

Theorem 9.1 *Let k be a perfect field of positive characteristic p. For each k-point x of $\mathcal{F}_n$ there is a k-point y of $\mathcal{F}\ell_n$ such that the height 1-neighbourhood of x is isomorphic to the height 1-neighbourhood of y times a smooth space by an isomorphism respecting stratifications.*

Proof We consider the de Rham cohomology H together with the Gauss–Manin connection on the height-1 neighborhood Y of x. We can trivialize H plus its Gauss–Manin connection on Y since the ideal of x has a divided power structure for which divided powers of degree $\geq p$ are zero. This implies that the orthogonal flags $E_\bullet$ and $G_\bullet$ on H are horizontal. We thus get a map from Y to the space of orthogonal flags on a standard orthogonal space, that is, an isomorphism from Y to a height-1 neighborhood on $\mathcal{F}\ell_n$. It is not difficult to see that it preserves strata. □

The following theorem is the main consequence of this. Denote the base space of $\mathcal{F}_n$ by $\mathcal{K}_n$.

Theorem 9.2 *The strata $\mathcal{U}_w$ possess the following properties:*

i) *Each stratum $\mathcal{U}_w$ is smooth of dimension $\ell(w)$.*

ii) *The closed stratum $\overline{\mathcal{U}}_w$ is reduced, Cohen–Macaulay and normal of dimension $\ell(w)$ and is the closure of $\mathcal{U}_w$ for all w in the Weyl group.*

iii) *If w is final then the restriction to $\mathcal{U}_w$ of the projection $\mathcal{F}_n \to \mathcal{K}_n$ to $\mathcal{U}_w$ is a finite surjective étale covering from $\mathcal{U}_w$ to $\mathcal{V}_w$ of degree equal to the number of final filtrations on a canonical filtration of type w.*

Proof This theorem follows from Thm 10.1 in exactly the same fashion as Corollary 8.4 in Section 8.2 of [10] follows from Theorem 8.1 there. □

In view of our calculations of the cycle classes, we need to know the degrees of the canonical projections $\pi_w : \mathcal{U}_w \to \mathcal{V}_w$ of our strata for final or twisted final elements. This degree is expressed as the number of ways we can put a final filtration on a canonical one. We now calculate these degrees.

Lemma 9.3 *Let $w \in W_m^B$ be a final element and let $\pi_w : \mathcal{U}_w \to \mathcal{V}_w$ be the restriction of the projection from $\mathcal{F}_n \to \mathcal{K}_n$ with $n = 2m + 1$.*

i) *For $1 \le k \le m-1$ we have* $\deg(\pi_{w_k})/\deg(\pi_{w_{k+1}}) = p^{2m-2k-1} + p^{2m-2k-2} + \cdots + 1$.
ii) *Similarly, we have* $\deg(\pi_{w_{m+k+1}})/\deg(\pi_{w_{m+k}}) = p^{2k-1} + p^{2k-2} + \cdots + 1$.

Proof Note that $\deg(\pi_w)$ is the number of final filtrations on a given canonical filtration of type w. For case i), we look at the number of lines in a linear space of dimension $2m - 2k$ over $\mathbb{F}_p$, i.e., the number of points in projective space of dimension $2m - 2k$. For case ii), we look at the number of isotropic lines in an orthogonal space of dimension $2k + 1$, i.e., the degree is the number of points on a quadric of dimension $2k - 1$. □

Lemma 9.4 *Let $w \in W_m^D$ be a final element and let $\pi_w : \mathcal{U}_w \to \mathcal{V}_w$ be the restriction of the projection from $\mathcal{F}_n \to \mathcal{K}_n$ with $n = 2m$.*

i) *For $1 \le k \le m-1$ we have* $\deg(\pi_{w_k})/\deg(\pi_{w_{k+1}}) = -p^{m-k-1} + \sum_{j=0}^{2m-2k-2} p^j$.
ii) *We have* $\deg(\pi_{w_{m-1}}) = \deg(\pi_{w_m}) = (\pi_{w_{m+1}}) = (\pi_{w_{m+2}}) = 1$.
iii) *Similarly, for $2 \le k \le m-1$ we have* $\deg(\pi_{w_{m+k+1}})/\deg(\pi_{w_{m+k}}) = p^{k-1} + \sum_{j=0}^{2k-2} p^j$.

Proof The proof is the same as for Lemma 9.3 except that the counts of isotropic lines are different. It also depends on whether the form is split or not but that is provided by Theorem 7.1. □

Lemma 9.5 *Let $w = w's'_m \in W_m'^D$ be a twisted final element and let $\pi_w : \mathcal{U}_w \to \mathcal{V}_w$ be the restriction of the projection from $\mathcal{F}_n \to \mathcal{K}_n$ with $n = 2m$.*

i) *For $1 \le k \le m-1$ we have* $\deg(\pi_{w_k})/\deg(\pi_{w_{k+1}}) = p^{m-k-1} + \sum_{j=0}^{2m-2k-2} p^j$.
ii) *We have* $\deg(\pi_{w_{m-1}}) = \deg(\pi_{w_m}) = (\pi_{w_{m+1}}) = (\pi_{w_{m+2}}) = 1$.
iii) *Similarly, for $2 \le k \le m-1$ we have* $\deg(\pi_{w_{m+k+1}})/\deg(\pi_{w_{m+k}}) = -p^{k-1} + \sum_{j=0}^{2k-2} p^j$.

Proof Again the proof is the same as for Lemma 9.3 with needed extra information provided by Theorem 7.1. □

10 Shuffles

In this section, we shall discuss an analogue and generalization of shuffles introduced in [10]. They play a role in describing maps between the strata $\mathcal{U}_w$ on the flag space and describe inseparable maps between strata. They are a key instrument for deciding whether the push forward of the corresponding cycle class will vanish. In fact, we saw that for a final stratum $\mathcal{U}_w$ the projection to $\mathcal{V}_w$ is finite étale. It will turn out that for a non-final stratum, the projection is lower-dimensional or factors through an inseparable map to a final stratum. These inseparable maps are described by shuffle maps as in [10]. We analyze the situation in detail.

We start with an elementary lemma on the length of elements in our Weyl groups. A general reference is [4].

Lemma 10.1 *The length satisfies the following properties.*

i) *Let w be an element of a Coxeter group and suppose that $\ell(ws_i) = \ell(w) - 1$. Then either $\ell(s_i w s_i) = \ell(w)$ or $\ell(s_i w s_i) = \ell(w) - 2$.*
ii) *For w an element of W_m^B, W_m^C or W_m^D and $1 \leq i < m$ we have that $\ell(ws_i) = \ell(w) - 1$ precisely when $w(i+1) < w(i)$ and then $\ell(s_i w s_i) = \ell(w) - 2$ precisely when $w^{-1}(i+1) < w^{-1}(i)$.*
iii) *For w an element of W_m^B we have $\ell(ws_m) = \ell(w) - 1$ precisely when $w(m) > w(m+1)$ and then $\ell(s_m w s_m) = \ell(w) - 2$ precisely when $w^{-1}(m+2) < m$.*
iv) *For w an element of W_m^C we have $\ell(ws_m) = \ell(w) - 1$ precisely when $w(m) > w(m+1)$ and then $\ell(s_m w s_m) = \ell(w) - 2$ precisely when $w^{-1}(m+1) < m$.*
v) *For $w \in W_m^D$ we have $\ell(ws_m) = \ell(w) - 1$ precisely when $w(m-1) > w(m+1)$ and $w(m) > w(m+2)$ and then $\ell(s_m w s_m) = \ell(w) - 2$ precisely when $w^{-1}(m+1) > w^{-1}(m-1)$ and $w^{-1}(m+2) > w^{-1}(m)$.*

Proof Easy. □

We now define the shuffle maps. Assume that the dimension of our orthogonal or symplectic space is n and that W_n denotes the Weyl group in question. Assume that $w \in W_n$ and that $\ell(ws_i) = \ell(w) - 1$ for some $1 < i \leq m$. This means that for the universal flags $\mathbb{E}_\bullet$ and $\mathbb{G}_\bullet$ on $\mathcal{U}_w$ the image of $G_{w(i+1)} \cap \mathbb{E}_{i+1}$ in $\mathbb{E}_{i+1}/\mathbb{E}_{i-1}$ is a line bundle. We define a new self-dual flag $\mathbb{E}'_\bullet$ on $\mathcal{U}_w$ by the condition that $\mathbb{E}'_j = \mathbb{E}_j$ for $i \neq j \leq n$ and $\mathbb{E}'_i/\mathbb{E}_{i-1}$ be equal to the image of $\mathbb{G}_{w(i+1)} \cap \mathbb{E}_{i+1}$. This then gives a map

$$\sigma_{w,i} : \mathcal{U}_w \to \mathcal{F}_n, \qquad (\mathbb{E}_\bullet, \mathbb{G}_\bullet) \mapsto (\mathbb{E}'_\bullet, \mathbb{G}_\bullet)$$

which we shall call the *i'th elementary shuffle map* for w. We say that the shuffle map is *unambiguous* if there is a $v \in W_n$ such that the image of $\sigma_{w,i}$ lies in $\mathcal{U}_v$. Please note the condition $i > 1$ which ensures that the first and last step of the Hodge filtration are left unchanged.

Proposition 10.2 *The shuffle map $\sigma_{w,i}$ satisfies the following properties.*

i) *The element $\sigma_{w,i}$ for $w \in W_n$ is unambiguous precisely when $\ell(s_i w s_i) = \ell(w)$. In that case the image of $\sigma_{w,i}$ is equal to $\mathcal{U}_{s_i w s_i}$ and $\sigma_{w,i}$ is finite and purely inseparable of degree p.*
ii) *If $\ell(s_i w s_i) = \ell(w) - 2$ then $\sigma_{w,i}$ maps onto $\mathcal{U}_{ws_i} \cup \mathcal{U}_{s_i w s_i}$. In particular it is not generically finite.*

Proof Assume first that $\ell(s_iws_i) = \ell(w)$. We may locally (in the étale topology) choose a basis adapted to the two flags, i.e., an orthonormal basis $e_1, \dots, e_n$ of M (on $\mathcal{U}_w$) such that $\mathbb{E}_j$ is spanned by $e_1, \dots, e_j$ and $\mathbb{G}_{w^{-1}(j)}$ is spanned by $e_{w^{-1}(1)}, \dots, e_{w^{-1}(j)}$. We then have that $\mathbb{E}'_i$ is spanned by $e_1, \dots, \hat{e}_i, e_{i+1}$. We may further assume that $C^{-1}e_j = e_{w^{-1}(j)} \bmod \mathbb{G}_{j-1}$ for all $1 < j < n$. Put $k := w^{-1}(i)$ and $\ell := w^{-1}(i+1)$, we then have, by the assumption and Lemma 10.1, that $k < \ell$. There is a λ such that $C^{-1}e_{i+1} = e_\ell + \lambda e_k \bmod \mathbb{G}_{i-1}$. We now put for $j \leq m$

$$e'_j = \begin{cases} e_j & \text{if } j \neq i, i+1, \ell, \\ e_{i+1} & \text{if } j = i, \\ e_i & \text{if } j = i+1, \text{ and} \\ e_\ell + \lambda e_k & \text{if } j = \ell. \end{cases}$$

By the assumption $k < \ell$ we get that $\mathbb{E}'_j$ is spanned by $e'_1, \dots, e'_j$, and we may extend it (uniquely) to an adapted basis for $\mathbb{E}'_\bullet$ and $\mathbb{G}'_\bullet$. This makes it clear that $(\mathbb{E}'_\bullet, \mathbb{G}'_\bullet)$ is of type s_iws_i. Consider conversely the universal flag $\mathbb{E}_\bullet$ on $\mathcal{U}_v$, where $v = s_iws_i$ and put again $k := v^{-1}(i)$ and $\ell := v^{-1}(i+1)$, where this time $k > \ell$ and $v^{-1}(i) < v^{-1}(i+1)$. We choose as before an adapted basis and let $\mathbb{E}'_i$ be spanned by $e_1, \dots, e_{i+1} + \rho e_i$ and then $\mathbb{G}'_i$ is spanned by $e_\ell + (\rho^p + \lambda)e_k$ and $\mathbb{G}_{i-1}$. It is then easy to see that $(\mathbb{E}'_\bullet, \mathbb{G}'_\bullet)$ is of type w precisely when $\rho^p + \lambda = 0$ which gives i).

We now assume that $\ell(s_iws_i) = \ell(w) - 2$. The setup is then the same as before except that we now have $k > \ell$. This means that $(\mathbb{E}'_\bullet, \mathbb{G}'_\bullet)$ will be of type ws_i if $\lambda \neq 0$ and of type s_iws_i if $\lambda = 0$. The converse is similar to the converse of i) (with the difference that the new flag pair will be of type w for all choices of ρ). ☐

Definition 10.3 A sequence of elements $w_1, \dots, w_r$ of W_n is said to be a *shuffle sequence* if for each $1 \leq k < r$ there is an $1 < i_k \leq m$ such that $\ell(w_ks_{i_k}) = \ell(w_k) - 1$ and $w_{k+1} = s_{i_k}w_ks_{i_k}$ with $\ell(s_{i_k}w_ks_{i_k}) = \ell(w_k)$. It is said be *ambiguous* if there is an i_r such that $\ell(w_rs_{i_r}) = \ell(w_r) - 1$ and $\ell(s_{i_r}w_ks_{i_r}) = \ell(w_r) - 2$, *final* if w_k is final, and *cyclic* if there are $1 \leq j < k \leq r$ such that $w_j = w_k$.

We shall now show that starting with a non-final element we always can find a shuffle sequence. Doing this, we see that we end up at a final stratum or we find that the image of our stratum under projection is lower dimensional. Recall that $\mathcal{K}_n$ denotes the base space of our flag space $\mathcal{F}_n$.

Proposition 10.4 *Shuffle sequences exist:*

i) *For every $w \in W_n$ there exists a shuffle sequence starting with w and which is either ambiguous, final, or cyclic.*

ii) *If there is an ambiguous or cyclic shuffle sequence starting with $w \in W_n$, then the restriction of the projection map $\mathcal{F}_n \to \mathcal{K}_n$ to $\mathcal{U}_w$ is not generically finite.*

iii) *Given a final shuffle sequence $w_1, \dots, w_r$, the restriction of the projection map $\mathcal{F}_n \to \mathcal{K}_n$ to $\mathcal{U}_{w_1}$ is the composite of a finite flat purely inseparable map of degree p^{r-1} and the finite étale map $\mathcal{U}_{w_r} \to \mathcal{V}_{w_r}$.*

Proof Let $w = w_1, \ldots, w_r$ be a shuffle sequence which is maximal for not being ambiguous, final, or cyclic. In particular, w_r is not final, and therefore, there is an $1 < i_r \leq m$ such that $\ell(w_r s_{i_r}) = \ell(w_r) - 1$. If $\ell(s_{i_r} w_r s_{i_r}) = \ell(w_r)$, then by the maximality we must have either that $w_{i_{r+1}} := s_{i_r} w_r s_{i_r}$ is final or appears in the sequence so that we get a final or cyclic sequence by adding $w_{i_{r+1}}$. If $\ell(s_{i_r} w_r s_{i_r}) = \ell(w_r) - 2$, we instead get an ambiguous sequence thus proving part i).

If there is an ambiguous sequence $w = w_1, \ldots, w_r$, then the projection map $\mathcal{U}_w$ factors by Proposition 10.2 as $\mathcal{U}_w \to \mathcal{U}_{w_r s_{i_r}} \cup \mathcal{U}_{s_{i_r} w_r s_{i_r}} \to \mathcal{K}_n$ and as $\ell(w_r s_{i_r}) < \ell(w)$ and $\ell(s_{i_r} w_r s_{i_r}) < \ell(w)$, and hence, $\mathcal{U}_w$ has an image of dimension smaller than that of $\mathcal{U}_w$. On the other hand, if there is a cyclic sequence, then the projection factors through an infinite sequence of $\sigma_{v,j}$'s and as each of them is of degree > 1 we get that image has lower dimension. This proves part ii).

Finally, assume that we have a final sequence $w = w_1, \ldots, w_r$. Then, the projection factors as the composite $\sigma_{w_{r-1}, i_{r-1}} \circ \cdots \circ \sigma_{w_1, i_1}$ and the projection $\mathcal{U}_{s_{i_r} w_r s_{i_r}} \to \mathcal{K}_n$. The latter is an étale cover of $\mathcal{V}_{w_r}$, and the first is finite purely inseparable of degree p. □

We shall call an ambiguous or cyclic shuffle a *degenerate shuffle*. Proposition 10.4 implies that either all shuffles of an element $w \in W_n$ are degenerate or they are all final. In the first case, the projection map restricted to $\overline{\mathcal{U}}_w$ is not generically finite on each of its irreducible components and in particular the image of $[\overline{\mathcal{U}}_w]$ is zero. In the second, the class of the push forward is nonzero and equal $p^\ell[\overline{\mathcal{U}}_v]$, where ℓ is the length of a final shuffle of w to the final element v.

11 Final elements

In order to calculate cycle classes of our strata, we shall apply a Pieri formula which gives an expression of the intersection product of a class of a stratum with a first Chern class in terms of cycle classes of strata of dimension one less. For this, we need a precise description of the colength one elements in the Weyl group below a given final (or twisted final) element and then determine whether these elements are degenerate or of shuffle type. In this auxiliary and rather technical section, we describe the elements involved. After treating the case of W_m^B in detail, we deal with the other cases more succinctly.

11.1 Final elements in W_m^B

We begin by factoring the final elements in the Weyl group W_m^B as a product of simple reflections.

Lemma 11.1 *The products $w_k = s_k s_{k+1} \cdots s_m s_{m-1} \cdots s_1$ with $1 \leq k \leq m-1$ and $w_{2m-k} = s_k s_{k-1} \cdots s_1$ with $m \geq k \geq 0$ are reduced expressions for the $2m$ final elements of W_m^B. We have $w_1 = w_\emptyset$ and $w_{2m} = 1$.*

Proof Easily verified. □

Note that the final elements are linearly ordered by the their length. We now determine the elements of colength 1 below a final element in the Bruhat order.

Proposition 11.2 *The elements of colength 1 below a final element of W_m^B in the Bruhat order are as follows:*

i) *The elements in W_m^B of colength 1 below the final element $w = s_k \cdots s_m \cdots s_1$ with $k < m$ are $s_k \cdots \hat{s}_i \cdots s_m \cdots s_1$ for $i = k, \dots, m-1$, and $s_k \cdots s_m \cdots \hat{s}_i \cdots s_1$ for $i = m-1, m-2, \dots, 1$. They are obtained from the final element by multiplying w to the right by the element s_α, where α is the root $\epsilon_1 + \epsilon_{k+1}$,...,$\epsilon_1 + \epsilon_m$, $\epsilon_1 - \epsilon_m$,...,$\epsilon_1 - \epsilon_2$, respectively.*
ii) *The elements of colength one below $w = s_m s_{m-1} \cdots s_1$ are the elements $s_m \cdots \hat{s}_i \cdots s_1$ for $i = m, \dots, 1$. They are obtained by multiplying w from the right by s_α where α is the root ϵ_1, $\epsilon_1 - \epsilon_m$, $\epsilon_1 - \epsilon_{m-1}, \dots, \epsilon_1 - \epsilon_2$, respectively.*
iii) *The elements of colength one below $w = s_k s_{k-1} \cdots s_1$ are the elements $s_k \cdots \hat{s}_i \cdots s_1$ for $i = k, \dots, 1$. They are obtained from the final element by multiplying w to the right by the element s_α, where α is the root $\epsilon_1 - \epsilon_{k+1}$,...,$\epsilon_1 - \epsilon_2$, respectively.*

Proof We know that the elements of colength 1 below an element are obtained by considering a reduced expression for the element, taking the elements obtained by removing one element from the expression, and then keeping the elements of colength 1. Lemma 11.1 provides a reduced expression. In case part i), among the elements obtained by removing one simple reflection from the reduced expression clearly the one obtained by removing s_m (when present) is not of colength 1 and the others are easily shown to be. Finally, if the element has the factorization $w' s_i w''$ and the colength 1 element has the factorization $w' w''$, then it is obtained by multiplying by $(w'')^{-1} s_i w''$ to the right , i.e., by s_α, where $\alpha = (w'')^{-1}(\alpha_i)$. From this, the rest follows by a simple calculation. The cases part ii) and part iii) are similar. □

We shall now consider the elements of colength 1 below a final element and determine if they have degenerate or final shuffle type.

Proposition 11.3 *The elements of colength 1 below a final element satisfy the following.*

i) *The element $s_k s_{k+1} \dots s_m \cdots \hat{s}_i \cdots s_1$ with $1 \leq i < k < m$ is degenerate.*
ii) *The element $s_k s_{k+1} \cdots s_m \cdots \hat{s}_i \cdots s_1$ with $m > i \geq k$ has an elementary shuffle to the element $s_k s_{k+1} \cdots s_m \cdots \hat{s}_{i+1} \cdots s_1$ if $i < m - 1$ and to $s_k s_{k+1} \cdots \hat{s}_{m-1} s_m \cdots s_1$ if $i = m - 1$.*
iii) *For $m > i > k$ the element $s_k \cdots \hat{s}_i \cdots s_m \cdots s_1$ has an elementary shuffle to the element $s_k \cdots \hat{s}_{i-1} \cdots s_m \cdots s_1$.*
iv) *The element $s_k \cdots \hat{s}_i \cdots s_1$ is degenerate if $i < k \leq m$.*

Proof Starting with part i) we note that the elements s_j with $k \leq j \leq i$ commute with the s_l with $i - 1 \geq l \geq 1$. This means that $s_k \cdots s_m \cdots \hat{s}_i \cdots s_1 = s_k \cdots s_m \cdots s_{i+2} s_{i-1} \cdots s_1 s_{i+1}$, and this implies that we can perform an $i + 1$'st shuffle giving the element $s_{i+1} s_k \cdots s_m \cdots s_{i+2} s_{i-1} \cdots s_1$. If $i + 1 = k$, this element has shorter length, while if $i+1=k-1$, we get $s_{k-1}\cdots s_m\cdots s_{i+2}s_{i-1}\cdots s_1$ and then move s_{i+2} to the right and perform a shuffle with it. We thus arrive at the element $s_ks_{k-1}s_k\cdots s_m\cdots s_{i+3}s_{i-1}\cdots s_1$, and by applying the braid relation, we get the element $s_{k-1}s_ks_{k-1}s_{k+1}\cdots s_m\cdots s_{i+3}s_{i-1}\cdots s_1$ and by moving s_{k-1}, the third factor, this is seen to equal $s_{k-1}s_k\cdots s_m\cdots s_{i+3}s_{i+1}s_{i-1}\cdots s_1$, and by moving $s_{i+1}=s_{k-1}$ to the right, then performing a shuffle by s_{i+1} we get an element of shorter length. If, however, $i+1<k-1$ we get $s_k\cdots s_m\cdots s_{i+3}s_{i+1}s_{i+2}s_{i-1}\cdots s_1$, and then, we can perform a shuffle by s_{i+2}. Continuing in this way, this leads to a shorter element.

We continue with part ii) and assume that $i+1\neq m$. Then, $s_ks_{k+1}\cdots s_m\ldots s_{i+1}s_{i-1}\cdots s_1$ equals $s_ks_{k+1}\cdots s_m\cdots s_{i+2}s_{i-1}\cdots s_1s_{i+1}$ as the involved simple transpositions commute. This means that we may perform an elementary shuffle to get the element $s_{i+1}s_ks_{k+1}\cdots s_m\cdots s_{i+2}s_{i-1}\cdots s_1$ which in turn is equal to $s_ks_{k+1}\cdots s_{i+1}s_is_{i+1}\cdots s_m\cdots s_{i+2}s_{i-1}\cdots s_1$. Using the braid rule, we get that this element equals $s_ks_{k+1}\cdots s_is_{i+1}s_i\cdots s_m\cdots s_{i+2}s_{i-1}\cdots s_1$, and we observe that this in turn equals $s_ks_{k+1}\cdots s_m\cdots s_{i+2}s_is_{i-1}\cdots s_1$ and this is $s_k\cdots s_m\hat{s}_{i+1}\cdots s_1$. By Lemma 11.1, this element is reduced so that we have performed an unambiguous shuffle to the claimed element. If instead $i+1=m$, we have that $s_ks_{k+1}\cdots s_ms_{m-2}\cdots s_1$ is equal to $s_ks_{k+1}\cdots s_{m-1}s_{m-2}\cdots s_1s_m$ which an elementary shuffle turns into $s_ms_ks_{k+1}\cdots s_{m-1}s_{m-2}\cdots s_1$ which on its turn equals the element $s_ks_{k+1}\cdots s_{m-2}s_ms_{m-1}s_{m-2}\cdots s_1$, a reduced expression of the right element.

For part iii) note that $s_k\cdots s_{i-1}s_{i+1}\ldots s_m\cdots s_1$ equals $s_k\cdots s_{i-2}s_{i+1}\cdots s_m\cdots s_{i-1}s_is_{i-1}\cdots s_1$ by moving s_{i-1} to the right, and by the braid rule equals $s_k\cdots s_{i-2}s_{i+1}\cdots s_m\cdots s_is_{i-1}s_i\cdots s_1$ in which the last s_i migrates to the right to give $s_k\cdots s_{i+1}\cdots s_m\cdots s_is_{i-1}\cdots s_1s_i$. Performing an elementary shuffle leads to $s_is_k\cdots s_{i+1}\cdots s_m\cdots s_is_{i-1}\cdots s_1$ which is equal to the element $s_k\cdots s_is_{i+1}\cdots s_m\cdots s_is_{i-1}\cdots s_1$. This is, still by the lemma, a reduced expression of the desired element. The proof of part iv) is analogous to that of part i). □

Corollary 11.4 *The non-degenerate elements of colength 1 below the final ones are as follows.*

i) *For a final element $w=s_k\ldots s_m\ldots s_1$ with $k<m$ the only non-degenerate elements of colength 1 below w are ws_α with $\alpha=\epsilon_1+\epsilon_{k+1},\ldots,\epsilon_1+\epsilon_m,\epsilon_1-\epsilon_m,\ldots,\epsilon_1-\epsilon_{k+1}$.*

ii) *For the final element $w=s_ms_{m-1}\ldots s_1$ there is only one non-degenerate element of colength 1 below it, namely ws_{ϵ_1}.*

iii) *For the final element $w=s_k\ldots s_1$ with $1\leq k\leq m-1$ there is only one non-degenerate element of colength one below w, namely $ws_{\epsilon_1-\epsilon_{k+1}}$.*

11.2 Final elements in W_m^D

This section is analogous to the preceding one, and we will therefore be brief.

Lemma 11.5 *The products $w_k=s_k\cdots s_{m-2}s_ms_{m-1}\cdots s_1$ with $1\leq k\leq m-2$ together with the product $w_{m-1}=s_ms_{m-1}\cdots s_1$, the elements $w_m=s_ms_{m-2}\cdots s_1$ and $w_{m+1}=s_{m-1}s_{m-2}\cdots s_1$ and the products $w_{m+j}=s_{m-j}s_{m-j-1}\cdots s_1$ with*

$j = 2, \ldots, m$ are reduced expressions for the $2m$ final elements of W_m^D. We have $w_1 = w_\emptyset$ and $w_{2m} = 1$.

Proof Easily verified. □

For each integer ℓ with $0 \leq \ell \leq 2m-2$ and $\ell \neq m-1$ there is one final element with length $\ell(w) = \ell$ while there are two final elements of length $m-1$. We can associate a graph with these $2m$ final elements by associating a vertex to each final element and an edge to a pair u, v if $v = s_j u$ for some s_j. Conjugation by s_m' interchanges the two final elements of length $m-1$.

$$\begin{tikzcd}
& & & & w_{m+1} \arrow[dr] & & & \\
w_1 \arrow[r] & w_2 \arrow[r] & \cdots \arrow[r] & w_{m-1} \arrow[ur] \arrow[dr] & & w_{m+2} \arrow[r] & \cdots \arrow[r] & w_{2m} \\
& & & & w_m \arrow[ur] & & &
\end{tikzcd}$$

We now turn to the colength 1 elements below the final elements.

Lemma 11.6 *The colength 1 elements below the final elements are as follows:*

i) *There are $2m-k-1$ elements in W_m^D of colength 1 below the final element $w = s_k \cdots s_{m-2} s_m \cdots s_1$ for $k \leq m-2$ and they are $s_k \cdots \hat{s}_i \cdots s_{m-2} s_m \cdots s_1 = w s_{\epsilon_1 + \epsilon_{i+1}}$ for $i = k, \ldots, m-2$, the elements $s_1 \cdots s_{m-2} \hat{s}_m s_{m-1} \cdots s_1 = w s_{\epsilon_1 + \epsilon_m}$, $s_1 \cdots s_{m-2} s_m \hat{s}_{m-1} \cdots s_1 = w s_{\epsilon_1 - \epsilon_m}$, and the elements $s_1 \cdots s_{m-2} s_m s_{m-1} \cdots \hat{s}_{m-i} \cdots s_1 = w s_{\epsilon_1 - \epsilon_{m+1-i}}$ for $i = 2, \ldots, m-1$.*

ii) *There are m elements of colength 1 below the final element $w = s_m s_{m-1} \ldots s_1$ and they are $w s_\alpha$ with $\alpha = \epsilon_1 + \epsilon_m, \epsilon_1 - \epsilon_m, \ldots, \epsilon_1 - \epsilon_2$.*

iii) *The elements in W_m^D of colength 1 below the final element $w = s_m s_{m-2} \cdots s_1$ are $s_{m-2} \cdots s_1 = w s_{\epsilon_1 + \epsilon_m}$ and $s_m s_{m-2} \cdots \hat{s}_i \cdots s_1 = w s_{\epsilon_1 - \epsilon_{i+1}}$ for $i = m-2, \ldots, 1$.*

iv) *The elements in W_m^D of colength 1 below the final element $w = s_{m-1} s_{m-2} \cdots s_1$ are $s_{m-2} \cdots s_1 = w s_{\epsilon_1 - \epsilon_m}$ and $s_{m-1} s_{m-2} \cdots \hat{s}_i \cdots s_1 = w s_{\epsilon_1 - \epsilon_{i+1}}$ for $i = m-2, \ldots, 1$.*

v) *The elements in W_m^D of colength 1 below the final element $w = s_k \cdots s_1$ with $1 \leq k \leq m$ are the elements $s_k \cdots \hat{s}_i \cdots s_1 = w s_{\epsilon_1 - \epsilon_{k+1}}$ for $i = k, \ldots, 1$.*

Proof The proof is analogous to the case of W_m^B treated in the preceding section. □

Again, we now consider the elements of colength 1 below a final element and determine if they have degenerate or final shuffle type.

Proposition 11.7 i) *The element $s_k \cdots s_{m-2} s_m \cdots \hat{s}_i \cdots s_1$ with $i < k \leq m-2$ is degenerate.*

ii) *The element $s_k \cdots s_{m-2} s_m \cdots \hat{s}_i \cdots s_1$ with $i \geq k$ has an elementary shuffle by (s_{i+1}) to the element $s_k \cdots s_{m-2} s_m \cdots \hat{s}_{i+1} s_i \cdots s_1$ if $i < m-2$ and a double shuffle (by $s_{m-1} s_m$) to $s_k \cdots \hat{s}_{m-2} s_m \cdots s_1$ if $i = m-2$ and $k < m-2$ and an elementary shuffle (by s_m) to $s_m \cdots s_1$ if $k = m-2 = i$.*

iii) *The elements* $s_k \cdots s_{m-2}s_m\hat{s}_{m-1}s_{m-2}\cdots s_1$ *and* $s_k \cdots s_{m-2}\hat{s}_m s_{m-1}\cdots s_1$ *are degenerate.*
iv) *For* $m-2 \geq i > k$ *the element* $s_k \cdots \hat{s}_i \cdots s_{m-2}s_m \cdots s_1$ *has an elementary shuffle (by* s_i*) to the element* $s_k \cdots \hat{s}_{i-1} \cdots s_{m-2}s_m \cdots s_1$.
v) *The element* $s_m \cdots \hat{s}_i \cdots s_1$ *is degenerate if* $m-1 \leq i \leq 1$.
vi) *The element* $s_m s_{m-2} \cdots \hat{s}_i \cdots s_1$ *with* $m-2 \geq i > 1$ *is degenerate.*

Proof The proof is analogous to the W_m^B case and is omitted. □

Corollary 11.8 *The non-degenerate elements of colength 1 below the final ones are as follows.*

i) *For a final element* $w = s_k \cdots s_{m-2}s_m \cdots s_1$ *with* $k \leq m-2$ *the only non-degenerate elements of colength 1 below w are* ws_α *with* $\alpha = \epsilon_1 + \epsilon_{k+1}, \ldots, \epsilon_1 + \epsilon_{m-1}, \epsilon_1 - \epsilon_{m-1}, \ldots, \epsilon_1 - \epsilon_{k+1}$.
ii) *For the final element* $s_m s_{m-1} \cdots s_1$ *there are two non-degenerate elements of colength 1 below it, namely* $ws_{\epsilon_1+\epsilon_m}$ *and* $ws_{\epsilon_1-\epsilon_m}$
iii) *For the final element* $w = s_m s_{m-2} \cdots s_1$ *there is only one non-degenerate element of colength 1 below it, namely* $ws_{\epsilon_1+\epsilon_m}$.
iv) *For the final element* $w = s_{m-1}s_{m-2} \cdots s_1$ *there is only one non-degenerate element of colength 1 below it, namely* $ws_{\epsilon_1-\epsilon_m}$.
v) *For the final element* $w = s_k \cdots s_1$ *with* $1 \leq k \leq m-2$ *there is only one non-degenerate element of colength 1 below it, namely* $ws_{\epsilon_1-\epsilon_{k+1}}$.

11.3 Twisted final elements in $W_m'^D$

The twisted final elements are of the form ws_m' with w as in Lemma 11.5. Similarly, the elements of colength 1 below a twisted final element ws_m' are of the form us_m' with u a colength 1 element below w as described in Lemma 11.6. We have to analyze whether these elements are degenerate or have final shuffle type. We omit the analogue of Proposition 11.7 and formulate immediately the analogue of Corollary 11.8

Corollary 11.9 *The non-degenerate elements of colength 1 below the final ones are as follows.*

i) *For a twisted final element* ws_m' *with* $w = s_k \cdots s_{m-2}s_m \cdots s_1$ *and* $k \leq m-2$ *the only non-degenerate elements of colength 1 below* ws_m' *are of the form* us_m' *with u equal to* ws_α *with* $\alpha = \epsilon_1 + \epsilon_{k+1}, \ldots, \epsilon_1 + \epsilon_m$*, and* $\epsilon_1 - \epsilon_m, \ldots, \epsilon_1 - \epsilon_{k+1}$.
ii) *For the twisted final element* ws_m' *with* $w = s_m s_{m-1} \cdots s_1$ *there are two non-degenerate elements of colength 1 below it, namely corresponding to* $ws_{\epsilon_1+\epsilon_m}$ *and* $ws_{\epsilon_1-\epsilon_m}$
iii) *For the twisted final element* ws_m' *with* $w = s_m s_{m-2} \cdots s_1$ *there is only one non-degenerate element of colength 1 below it, namely corresponding to* $ws_{\epsilon_1+\epsilon_m}$.
iv) *For the final element* ws_m' *with* $w = s_{m-1}s_{m-2} \cdots s_1$ *there is only one non-degenerate element of colength 1 below it, namely corresponding to* $ws_{\epsilon_1-\epsilon_m}$.
v) *For the final element* ws_m' *with* $w = s_k \cdots s_1$ *with* $1 \leq k \leq m-2$ *there is only one non-degenerate element of colength 1 below it, namely corresponding to* $ws_{\epsilon_1-\epsilon_{k+1}}$.

12 Pieri's formula and the cycle classes of the strata

Since the strata in our case, unlike the case of abelian varieties, are (almost) linearly ordered, we can fruitfully apply a Pieri type formula to get a formula for the classes of $\overline{\mathcal{V}}_\nu$. The appropriate formula is the Pieri formula of Pittie and Ram ([25]). There is a small problem in that the result only applies when we start with a G-torsor over a connected semi-simple group G, and in our case, the structure group is the disconnected group $\mathrm{O}(n)$. The resolution of this problem differs somewhat in the two cases of even or odd n so part of the discussion is postponed to the separate discussions for the two cases. In any case, the Pittie–Ram formula expresses the intersection product of the cycle class of a stratum with a first Chern class in terms of cycle classes of strata of one dimension less. We have to use the formula on the flag space and then project it down. The precise details of the Pieri formula differ enough between the odd and even-dimensional cases to make separate discussions in the two cases. Throughout we shall assume the versality assumption made in Sect. 9: We assume that we have a family $f : X \to S$ of N-marked K3 surfaces (where S may be an algebraic stack) such that S be smooth over $\mathbb{F}_p$ at all s and that the composed map $T_sS \to \mathrm{Hom}(H^0(X_s, \Omega^2_{X_s}), P)$ be surjective.

12.1 The odd-dimensional case

We now assume $n = 2m + 1$. Now, $\mathrm{O}(2m + 1) = \mathrm{SO}(2m + 1) \times \{\pm 1\}$ and hence an $\mathrm{O}(2m + 1)$-torsor is the same thing as one $\mathrm{SO}(2m + 1)$-torsor and one double cover. Thus, the problem mentioned above is resolved by considering instead the $\mathrm{SO}(2m + 1)$-torsor. In concrete terms, this means replacing our F-zip vector bundle H by $H \otimes \det(H)$.

We want to apply the Pieri formula to the two complete flags that we have on the flag space $\mathcal{F}_n$. If we let $\lambda = \sum_i n_i \ell_i$, where $\ell_i = c_1(E_i/E_{i-1})$ for $1 \le i \le m$ is the first Chern class corresponding to the root ϵ_i. The starting point for the Pieri formula is the following construction: Given a sequence $z = (z_1, \dots, z_m)$ of cohomology classes (of fixed degree) and a weight vector $\lambda = \sum_{i=1}^m n_i \epsilon_i$ (in the weight lattice of type B_m) we define $z^\lambda := \sum_i n_i z_i$. We shall apply this to $x = (\ell_1, \dots, \ell_m)$ and $y = (k_1, \dots, k_m)$, where $k_i := c_1(G_i/G_{i-1})$ and then, for suitable λ, we shall consider x^λ and $y^{w\lambda}$. However, the elements of x and of y span the same subgroup of the cohomology so we can also write $y^{w\lambda}$ as $x^{\lambda'}$ for a suitable λ'. Clearly, the association $\lambda \mapsto \lambda'$ is a linear operator on the weight lattice. It is easily seen that just as for the symplectic case it is given by $\lambda' = pw_\emptyset w(\lambda)$. From this point on, we shall only be considering elements of the form x^μ and for simplicity we shall write them just as μ. The Pieri formula (see the proof of [10, Thm 10.1] for details) now takes the form

$$(1 - pw_\emptyset w)(\lambda)[\overline{\mathcal{U}}_w] = -\sum_{\ell(ws_\alpha)=\ell(w)-1} \langle \alpha^\vee, \lambda \rangle [\overline{\mathcal{U}}_{ws_\alpha}].$$

The term $1 - pw_\emptyset w$ is viewed as an element of the group ring $\mathbb{Q}[W_m^B]$ acting on the roots ℓ_i, and the sum is over roots α such that the length $\ell(ws_\alpha)$ is one less than the length of w. Moreover, $\alpha^\vee$ is the usual coroot defined by α. To obtain a formula for the multiplication of $[\overline{\mathcal{U}}_w]$ by a given line bundle ρ, we have to solve the equation $(1 - pw_\emptyset w)(\lambda) = \rho$. If we put $v := w_\emptyset w$ and if we let c be the smallest positive integer such that $v^c(\rho) = s\rho$ for some $s \in \{\pm 1\}$, then a solution is given by

$$\lambda = \frac{1}{1 - sp^c} \sum_{i=0}^{c-1} p^i v^i(\rho).$$

We carry this out with $\rho = \ell_1 = \lambda_1$, the first Chern class of the Hodge bundle, such that we obtain a formula for $\lambda_1[\overline{\mathcal{U}}_w]$. We shall call c the *reduced orbit length* and say that the orbit is *even* or *odd* according to as s is $+1$ or -1. Then, we push down to the moduli space. The degenerate strata push down to zero and the non-degenerate to a power of p times the push down of a final stratum.

The final elements in this case are of the form $w_k = s_k \cdots s_m \cdots s_1$ with $1 \leq k < m$ and $w_{k+m} = s_{m-k} s_{m-k-1} \cdots s_1$ for $0 \leq k \leq m-1$ and $w_{2m} = 1$. Note that $w_\emptyset = w_1$ with this usage. The corresponding final strata on the flag space are $\overline{\mathcal{U}}_{w_k}$ with $k = 1, \ldots, 2m$ with corresponding strata $\overline{\mathcal{V}}_{w_k}$ on the moduli space. For a final element, we denote the canonical map $\overline{\mathcal{U}}_w \to \overline{\mathcal{V}}_w$ by π_w and its degree by $\deg(\pi_w)$.

Remark 12.1 The strata $\overline{\mathcal{V}}_{w_k}$ for $1 \leq k \leq m$ are the strata corresponding to finite height equal to $\geq k$, the stratum $\overline{\mathcal{V}}_{w_{m+1}}$ is the supersingular stratum and the stratum $\overline{\mathcal{V}}_{w_{k+m}}$ corresponds to Artin invariant $\leq m + 1 - k$ for $1 \leq k \leq m$.

Theorem 12.2 *The cycle classes of the final strata $\overline{\mathcal{V}}_w$ on the base S are powers of λ_1 times polynomials in p given by*

$$\text{i)}\quad [\overline{\mathcal{V}}_{w_k}] = (p-1)(p^2-1)\cdots(p^{k-1}-1)\lambda_1^{k-1} \quad \textit{if } 1 \leq k \leq m,$$

$$\text{ii)}\quad [\overline{\mathcal{V}}_{w_{m+1}}] = \frac{1}{2}(p-1)(p^2-1)\cdots(p^m-1)\lambda_1^m,$$

$$\text{iii)}\quad [\overline{\mathcal{V}}_{w_{m+k}}] = \frac{1}{2}\frac{(p^{2k}-1)(p^{2(k+1)}-1)\cdots(p^{2m}-1)}{(p+1)\cdots(p^{m-k+1}+1)}\lambda_1^{m+k-1} \quad \textit{if } 2 \leq k \leq m.$$

Proof We start with a final element w of the form $w_k = s_k \cdots s_m \cdots s_1$ with $1 \leq k < m$. The colength 1 elements $w_k s_\alpha$ that are not degenerate correspond to the $2m - 2k$ elements $\alpha_1 = \epsilon_1 + \epsilon_{k+1}, \ldots, \alpha_{m-k} = \epsilon_1 + \epsilon_m, \alpha_{m-k+1} = \epsilon_1 - \epsilon_m, \ldots, \alpha_{2m-2k} = \epsilon_1 - \epsilon_{k+1}$. These are the only elements that will contribute to the push down. Note that we have $w_k s_{\alpha_1} = w_{k+1}$, again a final element. For the element $v = w_\emptyset w$ we have $v^j(1) = 2m + 1 - k + j$ for $j = 1, \ldots, k-1$ which means that the reduced orbit length is k and the orbit even. We thus find that

$$(1 - pv)\lambda_1 = \sum_{i=0}^{k-1} p^i v^i(\ell_1) = \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}.$$

Therefore, the Pieri formula gives

$$\begin{aligned}(p^k-1)\lambda_1[\overline{\mathcal{U}}_{w_k}] \equiv {} & \sum_{j=1}^{m-k}\left(\epsilon_1+\epsilon_{k+j}, \ell_1-\sum_{i=1}^{k-1}p^i\ell_{k+1-i}\right)[\overline{\mathcal{U}}_{ws_{\alpha_j}}] \\ & + \sum_{j=1}^{m-k}\left(\epsilon_1-\epsilon_{m+1-j}, \ell_1-\sum_{i=1}^{k-1}p^i\ell_{k+1-i}\right)[\overline{\mathcal{U}}_{ws_{\alpha_{m-k+j}}}],\end{aligned}$$

where $\equiv$ means that we count modulo degenerate strata. Pushing it down annihilates the classes of the degenerate strata because these loose dimension and yields

$$\begin{aligned}(p^k-1)\lambda_1[\overline{\mathcal{V}}_{w_k}]\deg(\pi_{w_k}) &= \sum_{j=1}^{2m-2k}[\overline{\mathcal{V}}_{w_{k+1}}]\deg(\pi_{ws_{\alpha_j}}) \\ &= (1+p+\cdots+p^{2m-2k-1})[\overline{\mathcal{V}}_{ws_{k+1}}]\deg(\pi_{w_{k+1}})\end{aligned}$$

since the $w_k s_{\alpha_j}$ for $j=2,\ldots,2m-2k$ are shuffles of $w_{k+1}=ws_{\alpha_1}$ which map to $\overline{\mathcal{V}}_{w_{k+1}}$ with degree p^{j-1}. By Lemma 9.3 we have $\deg(\pi_{w_k})/\deg(\pi_{w_{k+1}}) = p^{2m-2k-1}+\cdots+1$ and get $[\overline{\mathcal{V}}_{w_{k+1}}]=(p^k-1)[\overline{\mathcal{V}}_{w_k}]$ for $k=1,\ldots,m-1$. Since $[\overline{\mathcal{V}}_{w_1}]=1$ part i) follows.

For part ii), we note that there is only one non-degenerate element of colength 1, namely $ws_\alpha = w_{m+1}$ and it corresponds to $\alpha=\epsilon_1$ with $\alpha^\vee = 2\epsilon_1$. Note that $v=[m+2, 2m+1, 2, 3, \ldots, m-1]$ and $\sum_{i=0}^{m-1}p^i v^i(\ell_1) = \ell_1-\sum_{i=1}^{m-1}p^i\ell_{m+1-i}$. This gives

$$(p^m-1)[\overline{\mathcal{V}}_{w_m}]\deg(\pi_{w_m}) = 2\,[\overline{\mathcal{V}}_{w_{m+1}}]\deg(\pi_{w_{m+1}})$$

and we observe that $\deg(\pi_{w_m})=1=\deg(\pi_{w_{m+1}})$. This proves ii). For the case iii), we consider a final element $w_{k+m}=s_{m-k}s_{m-k-1}\cdots s_1$ with $k\geq 1$. There is only one non-degenerate element $w_{k+m}s_\alpha$ of colength 1, namely w_{k+1+m} with $\alpha=\epsilon_1-\epsilon_{m+1-k}$.

The element $v=w_\emptyset w_{m+k}=[m, 2m+1, 2, 3, \ldots]$ has an odd orbit of reduced orbit length $m+1-k$ and thus $v^{m+1-k}\lambda_1=-\lambda_1$ so that $\sum_{i=0}^{m-k}p^i v^i\ell_1 = \ell_1 + \sum_{i=1}^{m-k}p^i\ell_{m+2-k-i}$ and the Pieri formula gives

$$(p^{m-k}+1)\lambda_1[\overline{\mathcal{V}}_{m+k}]\deg(\pi_{w_{m+k}}) = (p-1)[\overline{\mathcal{V}}_{w_{m+k+1}}]\deg(\pi_{w_{m+k+1}}).$$

Here, we have $\deg(\pi_{w_{m+k+1}})/\deg(\pi_{w_{m+k}}) = p^{2k-1}+\cdots+1$ which gives $(p^{m-k}+1)\lambda_1[\overline{\mathcal{V}}_{m+k}] = (p^{2k}-1)[\overline{\mathcal{V}}_{w_{m+k+1}}]$. This proves the formulas.

That the formulas are up to a factor 1/2 polynomials in $\mathbb{Z}[\lambda_1, p]$ is clear for cases i) and ii) and follows from the next remark for case iii). ◻

Remark 12.3 The formula for case iii) can also be written as

$$[\overline{\mathcal{V}}_{w_{m+k}}] = \frac{1}{2}\left(\prod_{j=1}^{m+1-k}(p^j-1)\right)\begin{bmatrix} m \\ m+1-k \end{bmatrix}_{p^2}\lambda_1^{m+k-1},$$

where $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q$ is the usual q-binomial coefficient.

Remark 12.4 Theorem 1.1 in the Introduction is the special case where we take S equal to the moduli space of polarized K3 surfaces of degree d, prime to p and where $m = 10$. The versality condition is verified in a standard way (and essentially the same as the proof for the case of elliptic K3 surfaces with a section below).

12.2 The even-dimensional case

The reduction to an SO-torsor in the even case is more involved than in the odd case. To begin with if we have an $\mathrm{O}(2m)$-torsor $P \to X$ we get a double cover $Y := P/\mathrm{SO}(2m) \to X$ and the quotient map $P \to Y$ is an $\mathrm{SO}(2m)$-torsor. However, in order to have a Bruhat cell decomposition of $P \to Y$ (which is necessary even to formulate the Pieri formula) we need a reduction of the structure group to B (a Borel subgroup of $\mathrm{SO}(2m)$). This we get from our original setup in the following way: We can find a subgroup $B' \subset \mathrm{O}(2m)$ containing B as a subgroup of index 2 and we assume that we have a B'-torsor $Q \to X$ which then gives rise to a B-torsor $Q \to Q/B = Y$. If we look at the corresponding G/B-fibrations (where $G = \mathrm{SO}(2m)$) we get a commutative diagram

$$\begin{tikzcd} Q \times_B G/B \arrow[r] \arrow[d] & Q \times_{B'} G/B \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

and we get a Pieri formula to $Q \times_B G/B \to Y$ and then push it down to $Q \times_{B'} G/B$. What happens during this push down is the following: The class λ_1 (which is the only class for which we shall use the Pieri formula) is the pullback of a class on $Q \times_{B'} G/B$ so by the projection formula it can be moved out of the push down. We get a "Bruhat decomposition" also of $Q \times_{B'} G/B$, but the strata now corresponds to B'-orbits of G/B. Such an orbit is a union of one or two B-orbits depending on whether or not an element in $B' \setminus B$ fixes the B-orbit or not. Hence, the projection $Q \times_B G/B \to Q \times_{B'} G/B$ maps Bruhat strata to Bruhat strata and two strata $\overline{\mathcal{U}}_w$ and $\overline{\mathcal{U}}_{w'}$ in $Q \times_B G/B$ are mapped to the same stratum precisely when either $w' = w$ or $w' = s'_m w s'_m$. In our specific case, the B'-bundle arises by starting with a G'-torsor ($G' = \mathrm{O}(2m)$) $P \to X$ and then pulling it back along $P \times_{G'} G'/B'$ (note that $G'/B' = G/B$) where this pullback has a canonical reduction of its structure group to B'.

In flag terms, we have the following description. Let E be a quadratic vector bundle of rank $2m$ over X. The pairing on it induces an isomorphism $\det(E) \otimes \det(E) \xrightarrow{\sim} \mathcal{O}_X$

and hence gives a double cover $Y \to X$. We can also consider the *almost complete* flag space $\mathcal{F} \to X$ of self-dual flags $0 \subset E_1 \subset E_2 \subset \cdots \subset E_{m-1} \subset E_{m+1} \subset \cdots \subset E_{2m} = E$ with $\dim E_i = i$. The fiber product $\mathcal{F}' := Y \times_X \mathcal{F}$ has the explicit description as the space of complete self-dual flags $0 \subset E_1 \subset E_2 \subset \cdots \subset E_{m-1} \subset E_m \subset E_{m+1} \subset \cdots \subset E_{2m} = E$, and the double cover involution on Y induces the operation on such flags which replaces E_m by the other totally isotropic m-dimensional subspace $E_{m-1} \subset E'_m \subset E_{m+1}$. The fiber product $\mathcal{F}'' := \mathcal{F}' \times_X \mathcal{F}'$ consisting of pairs $(E_\bullet, F_\bullet)$ of complete self-dual flags split up in two components: one is $\mathcal{F}''_0$, where $\dim(E_m \cap F_m) \equiv m \bmod 2$, and the other one is $\mathcal{F}''_1$, where $\dim(E_m \cap F_m) \equiv m + 1 \bmod 2$. The group $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts on $\mathcal{F}' \times_X \mathcal{F}'$, a group factor acting on the corresponding factor of $\mathcal{F}' \times_X \mathcal{F}'$. The elements $(1, 0)$ and $(0, 1)$ permutes the two components and $(1, 1)$ preserves them. Each element $w \in W'^D_m$ gives a stratum of $\mathcal{F}''$ consisting of the flags in relative position w. When $w \in W^D_m$, the stratum lies in $\mathcal{F}''_0$ and when $w \in W^D_m s'_m$ it lies in $\mathcal{F}''_1$. The group element $(1, 0)$ then takes the stratum of w to that of ws'_m and the element $(0, 1)$ takes w to that of $s'_m w$.

12.2.1 The untwisted even case

The first step in getting to a Pieri formula is to identify the linear map that takes λ to λ'. This time it is *not* given by $pw_\emptyset w$ as $w_\emptyset(m) = m + 1$ which does not have the desired effect. Instead, we have to use the linear map $pw'_\emptyset w$ because $w'_\emptyset(m) = m$. This means that Pieri's formula takes the form

$$(1 - pw'_\emptyset w)(\lambda)[\overline{\mathcal{U}}_w] = - \sum_{\ell(ws_\alpha)=\ell(w)-1} \langle \alpha^\vee, \lambda\rangle [\overline{\mathcal{U}}_{ws_\alpha}]$$

Recall that the final elements are the $2m$ elements $w_k = s_k \cdots s_{m-2} s_m \cdots s_1$ for $k = 1, \ldots, m-2$, $w_{m-1} = s_m s_{m-1} \cdots s_1$, $w_m = s_m s_{m-2} \cdots s_1$ and $w_{m+j} = s_{m-j} \cdots s_1$ for $j = 1, \ldots, m-1$ and $w_{2m} = 1$. Moreover, there is an automorphism of W^D_m interchanging w_m and w_{m+1} given by conjugation by s'_m.

Theorem 12.5 *The cycle classes of the final strata $\overline{\mathcal{V}}_w$ for final $w_j \in W^D_m$ on the base S are powers of λ_1 times polynomials in p given by*

i) $[\overline{\mathcal{V}}_{w_k}] = (p-1)(p^2-1)\cdots(p^{k-1}-1)\lambda_1^{k-1}$ *if* $k \le m-1$,

ii) $[\overline{\mathcal{V}}_{w_{m+1}}] = (p-1)(p^2-1)\cdots(p^{m-1}-1)\lambda_1^{m-1}$,

iii) $[\overline{\mathcal{V}}_{w_{m+k}}] = \dfrac{1}{2}\dfrac{\prod_{i=1}^{m-1}(p^i-1)\prod_{i=m-k+2}^{m}(p^i+1)}{\prod_{i=1}^{k-2}(p^i+1)\prod_{i=1}^{k-1}(p^i-1)}\lambda_1^{m+k-2}$ *if* $2 \le k \le m$.

Furthermore, we have that $\overline{\mathcal{V}}_{w_m} = \emptyset$.

Proof Let $w_k = s_k \ldots s_{m-2} s_m \ldots s_1$ be a final element with $1 \le k \le m-2$. There are $2m - 2k - 2$ non-degenerate elements of colength 1 under w_k and they are of the form ws_α with $\alpha_1 = \epsilon_1 + \epsilon_{k+1}, \ldots, \alpha_{m-k-1} = \epsilon_1 + \epsilon_{m-1}$ and $\alpha_{m-k} = \epsilon_1 -$

$\epsilon_{m-1}, \dots, \alpha_{2m-2k-2} = \epsilon_1 - \epsilon_{k+1}$. We find that $v := w'_\emptyset w_k$ has an even orbit of reduced orbit length k and

$$(1 - pv)\lambda_1 = \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}.$$

Therefore, the Pieri formula gives

$$\begin{aligned}(p^k - 1)\lambda_1[\overline{\mathcal{U}}_{w_k}] \equiv \sum_{j=1}^{m-k-1} &\left(\epsilon_1 + \epsilon_{k+j}, \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}\right)[\overline{\mathcal{U}}_{ws_{\alpha_j}}] \\ &+ \sum_{j=m-k}^{2m-2k-2} \left(\epsilon_1 - \epsilon_{2m-k-1-j}, \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}\right)[\overline{\mathcal{U}}_{ws_{\alpha_j}}],\end{aligned}$$

where $\equiv$ means again that we work modulo degenerate strata. Pushing it down annihilates the classes of the degenerate strata and yields

$$\begin{aligned}(p^k - 1)\lambda_1[\overline{\mathcal{V}}_{w_k}] \deg(\pi_{w_k}) &= \sum_{j=1}^{2m-2k-2} [\overline{\mathcal{V}}_{w_{k+1}}] \deg(\pi_{ws_{\alpha_j}}) \\ &= (1 + \cdots + p^{m-k-2} + p^{m-k} + \cdots + p^{2m-2k-2})[\overline{\mathcal{V}}_{w_{k+1}}] \deg(\pi_{w_{k+1}}),\end{aligned}$$

since the $w_k s_{\alpha_j}$ for $j = 1, \dots, m-k-1$ are shuffles of $w_{k+1} = ws_{\alpha_0}$ for which $\overline{\mathcal{U}}_{w_k s_{\alpha_j}}$ maps to $\overline{\mathcal{U}}_{w_{k+1}}$ with degree p^{j-1}, while for $j = m-k, \dots, 2m-2k-2$ we get degree p^j. By Lemma 9.3 we have $\deg(\pi_{w_k})/\deg(\pi_{w_{k+1}}) = p^{2m-2k-2} + \cdots + p^{m-k} + p^{m-k-2} + \cdots + 1$ and hence get $[\overline{\mathcal{V}}_{w_{k+1}}] = (p^k - 1)\lambda_1[\overline{\mathcal{V}}_{w_k}]$ for $k = 1, \dots, m-1$. Since $[\overline{\mathcal{V}}_{w_1}] = 1$ part i) follows.

For part ii), we consider the final element $w = s_m s_{m-1} \cdots s_1$ and see that $v := w'_\emptyset w$ has an even orbit of reduced orbit length $m-1$ and that $(1 - pv)\lambda_1 = \ell_1 - \sum_{i=1}^{m-2} p^i \ell_{m-i}$. In this case, there are two non-degenerate elements of colength 1 namely ws_α with α being equal to $\epsilon_1 + \epsilon_m$ and $\epsilon_1 - \epsilon_m$ respectively. Applying the Pieri formula and pushing down first to the unoriented flag space and then to the moduli space, we get

$$(p^{m-1} - 1)\lambda_1[\overline{\mathcal{V}}_{m-1}] \deg(\pi_{w_{m-1}}) = [\overline{\mathcal{V}}_{w_m + w_{m+1}}] \deg(\pi_{w_m}).$$

By Lemma 9.3 $\deg(\pi_{w_{m-1}}) = \deg(\pi_{w_m}) = 1$ which gives ii) after pushing down.

For part iii), we consider the element $w_m = s_m s_{m-2} \cdots s_1$. The element $v := w'_\emptyset w$ has an odd orbit of reduced orbit length m and we have

$$(1 - pv)\lambda_1 = \ell_1 - p\ell_m + \sum_{i=2}^{m-1} p^i \ell_{m+1-i}.$$

There is now only one non-degenerate element of colength 1, namely $w_m s_\alpha$ with $\alpha = \epsilon_1 + \epsilon_m$. We get

$$(p^m + 1)\lambda_1[\overline{\mathcal{V}}_{w_m + w_{m-1}}]\deg(\pi_{w_m}) = (p-1)[\overline{\mathcal{V}}_{w_{m+2}}]\deg(\pi_{w_{m+2}}).$$

Again by Lemma 9.4 we have $\deg(\pi_{w_m}) = \deg(\pi_{w_{m+2}}) = 1$. Now, take $w = w_{m+j} = s_{m-j}\cdots s_1$ with $j \geq 2$. The element $v := w'_\emptyset w$ has an odd orbit of reduced orbit length $m - j + 1$. We get

$$(1 - pv)\lambda_1 = \ell_1 + \sum_{i=1}^{m-j} p^i \epsilon_{m+2-j-i}.$$

There is again only one non-degenerate element ws_α with $\alpha = \epsilon_1 - \epsilon_{m+1-j}$. Therefore, $\langle \alpha^\vee, \lambda\rangle = (1 - p^{j-1})$. We find

$$(p^{m+1-j} + 1)\lambda_1[\overline{\mathcal{V}}_{w_{m+j}}]\deg(\pi_{w_{m+j}}) = (p^{j-1} - 1)[\overline{\mathcal{V}}_{w_{m+j}}]\deg(\pi_{w_{m+j+1}}).$$

By Lemma 9.4 we have $\deg(\pi_{w_{m+j+1}})\deg(\pi_{w_{m+j}}) = p^{2j-2} + \cdots + 2p^{j-1} + \cdots + 1$. Using the factorization $p^{2j-2} + \cdots + 2p^{j-1} + \cdots + 1 = (p^{j-1} + 1)(p^{j-1} + p^{j-2} + \cdots + 1) = (1 + p^{j-1})/(1 - p^j)$ and iterating we get the formula. As there is only a small number cases ($m \leq 10$), the fact that one gets polynomials is most easily verified by explicit computation (one could also use [10, Prop. 13.2]). □

Remark 12.6 Similarly to the odd case we can write the third formula as

$$\frac{p^{k-1} + 1}{p^m - 1}\left(\prod_{i=1}^{m+1-k}(p^i - 1)\right)\begin{bmatrix} m \\ k-1 \end{bmatrix}_{p^2},$$

though this does not make it visibly a polynomial in p.

12.2.2 The twisted even case

We now turn to the twisted even-dimensional case. Going back to the previous notation, we have the space $\mathcal{F}''$ of pairs of complete flags of H where H now is the de Rham cohomology of the universal K3 surface over our moduli space. We have a disjoint decomposition $\mathcal{F}'' = \mathcal{F}''_0 \sqcup \mathcal{F}''_1$, where $\mathcal{F}''_0$ is the G/B-fibration over $\mathcal{F}'$ with structure group B and where we consequently have a Pieri formula. However, the F-zip structure on H gives a section of the projection (on the first factor) $\mathcal{F}'' \to \mathcal{F}'$ which is contained completely in $\mathcal{F}''_1$. In order to get a section along which we can pullback a Pieri formula on $\mathcal{F}''_0$, we must compose with the isomorphism $\mathcal{F}''_1 \xrightarrow{\sim} \mathcal{F}''_0$ obtained by applying the involution of $\mathcal{F}'$ acting on the second factor (say) of $\mathcal{F}''$. This extra involution implies that the linear map $\lambda \mapsto \lambda'$ is now given by $pw_\emptyset w$ (and *not* by $pw'_\emptyset w$ as in the untwisted case). Apart from that the argument of the Pieri formula proceeds along lines very similar to the untwisted case.

Recall that the final elements are the $2m$ elements $w_k = s_k \cdots s_{m-2}s_m \cdots s_1$ for $k = 1, \dots, m-2$, $w_{m-1} = s_m s_{m-1} \cdots s_1$, $w_m = s_m s_{m-2} \cdots s_1$ and $w_{m+j} = s_{m-j} \cdots s_1$ for $j = 1, \dots, m-1$ and $w_{2m} = 1$.

Theorem 12.7 *The cycle classes of the final strata $\overline{\mathcal{V}}_w$ for twisted final elements $w_j \in W_m^D s'_m$ on the base S are powers in λ_1 with coefficients that are polynomials in p given by*

$$\text{i)}\quad [\overline{\mathcal{V}}_{w_k}] = (p-1)(p^2-1)\cdots(p^{k-1}-1)\lambda_1^{k-1} \quad \textit{if} \ \ k \le m-1,$$

$$\text{ii)}\quad [\overline{\mathcal{V}}_{w_m}] = (p-1)(p^2-1)\cdots(p^m-1)\lambda_1^{m-1},$$

$$\text{iii)}\quad [\overline{\mathcal{V}}_{w_{m+k}}] = \frac{1}{2}\frac{\prod_{i=1}^{m}(p^i-1)\prod_{i=m-k+2}^{m-1}(p^i+1)}{\prod_{i=1}^{k-1}(p^i+1)\prod_{i=1}^{k-2}(p^i-1)}\lambda_1^{m+k-2} \quad \textit{if} \ \ 2 \le k \le m.$$

Furthermore, we have $\overline{\mathcal{V}}_{w_{m+1}} = \emptyset$.

Proof Let $w_k = s_k \cdots s_{m-2}s_m \cdots s_1$ be a final element with $1 \le k \le m-2$. There are $2m-2k$ non-degenerate elements of colength 1 under w_k, and they are of the form ws_α with $\alpha_1 = \epsilon_1 + \epsilon_{k+1}, \dots, \alpha_{m-k} = \epsilon_1 + \epsilon_m$ and $\alpha_{m-k+1} = \epsilon_1 - \epsilon_m, \dots, \alpha_{2m-2k} = \epsilon_1 - \epsilon_{k+1}$. We find that $v := w_\emptyset w_k$ has an even orbit of reduced orbit length k and

$$(1-pv)\lambda_1 = \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}$$

Therefore, the Pieri formula gives

$$\begin{aligned}(p^k-1)\lambda_1[\overline{\mathcal{U}}_{w_k}] \equiv{}& \sum_{j=0}^{m-k}\left(\epsilon_1 + \epsilon_{k+j}, \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}\right)[\overline{\mathcal{U}}_{ws_{\alpha_j}}] \\ &+ \sum_{j=m-k+1}^{2m-2k}\left(\epsilon_1 - \epsilon_{2m-k+1-j}, \ell_1 - \sum_{i=1}^{k-1} p^i \ell_{k+1-i}\right)[\overline{\mathcal{U}}_{ws_{\alpha_{-j}}}],\end{aligned}$$

where $\equiv$ means again that we work modulo degenerate strata. Pushing it down annihilates the classes of the degenerate strata and yields

$$\begin{aligned}(p^k-1)\lambda_1[\overline{\mathcal{V}}_{w_k}]\deg(\pi_{w_k}) &= \sum_{j=1}^{2m-2k}[\overline{\mathcal{V}}_{w_{k+1}}]\deg(\pi_{ws_{\alpha_j}}) \\ &= (1 + \cdots + 2p^{m-k-1} + \cdots + p^{2m-2k-2})[\overline{\mathcal{V}}_{w_{k+1}}]\deg(\pi_{w_{k+1}}),\end{aligned}$$

since the $w_k s_{\alpha_j}$ for $j = 1, \dots, m-k$ are shuffles of $w_{k+1} = ws_{\alpha_{k+1}}$ for which $\overline{\mathcal{U}}_{w_k s_{\alpha_j}}$ maps to $\overline{\mathcal{U}}_{w_{k+1}}$ with degree p^{j-1}, while for $j = m-k+1, \dots, 2m-2k$ we get degree p^{j-2}. By Lemma 9.4 we have $\deg(\pi_{w_k})/\deg(\pi_{w_{k+1}}) = p^{2m-2k-2} + \cdots + 2p^{m-k-1} + \cdots + 1$ and hence get $[\overline{\mathcal{V}}_{w_{k+1}}] = (p^k-1)[\overline{\mathcal{V}}_{w_k}]$ for $k = 1, \dots, m-1$. Since

$[\overline{\mathcal{V}}_{w_1}] = 1$ part i) follows. For part ii), we consider the final element $w = s_m s_{m-1} \cdots s_1$ and see that $v := w_\emptyset w$ has an even orbit of reduced orbit length $m - 1$ and that $(1 - pv)\lambda_1 = \ell_1 - \sum_{i=1}^{m-2} p^i \ell_{m-i}$. In this case, there are two non-degenerate elements of colength 1 namely ws_α with $\alpha_1 = \epsilon_1 + \epsilon_m$ and $\alpha_2 = \epsilon_1 - \epsilon_m$. Applying the formula we get

$$(p^{m-1} - 1)\lambda_1 [\overline{\mathcal{V}}_{m-1}] \deg(\pi_{w_{m-1}}) = [\overline{\mathcal{V}}_{ws_{\alpha_1}}] \deg(\pi_{w_m}) + [\overline{\mathcal{V}}_{ws_{\alpha_2}}] \deg(w_{m+1}).$$

By Lemma 9.4 $\deg(\pi_{w_{m-1}}) = \deg(\pi_{w_m}) = 1$ which gives ii) after pushing down.

For part iii), we consider the element $w_m = s_m s_{m-2} \cdots s_1$. The element $v := w_\emptyset w$ has an even orbit of reduced orbit length m and we have

$$(1 - pv)\lambda_1 = \ell_1 + p\ell_m - \sum_{i=2}^{m-1} p^i \ell_{m+1-i}.$$

There is now only one non-degenerate element of colength 1, namely $w_m s_\alpha$ with $\alpha = \epsilon_1 + \epsilon_m$. We get

$$(p^m - 1)\lambda_1 [\overline{\mathcal{V}}_{w_m}] \deg(\pi_{w_m}) = (p + 1)[\overline{\mathcal{V}}_{w_{m+1}}] \deg(\pi_{w_{m+1}}).$$

Again by Lemma 9.4 we have $\deg(\pi_{w_m}) = \deg(\pi_{w_{m+2}}) = 1$.

Now, take $w = w_{m+j} = s_{m-j} \cdots s_1$ with $j \geq 2$. The element $v := w_\emptyset w$ has an odd orbit of reduced orbit length $m - j + 1$. We thus get

$$(1 - pv)\lambda_1 = \ell_1 + \sum_{i=1}^{m-j} p^i \epsilon_{m+2-j-i}.$$

There is again only one non-degenerate element ws_α with $\alpha = \epsilon_1 - \epsilon_{m+1-j}$. Therefore, $\langle \alpha^\vee, \lambda \rangle = (1 - p^{j-1})$. We find

$$(p^{m+1-j} + 1)\lambda_1 [\overline{\mathcal{V}}_{w_{m+j}}] \deg(\pi_{w_{m+j}}) = (p^{j-1} - 1)[\overline{\mathcal{V}}_{w_{m+j}}] \deg(\pi_{w_{m+j+1}}).$$

By Lemma 9.4 we have $\deg(\pi_{w_{m+j+1}}) = p^{2j-2} + \cdots + p^j + p^{j-2} + \cdots + 1$. Using the factorization $p^{2j-2} + \cdots + p^j + p^{j-2} + \cdots + 1 = (p^j + 1)(p^{j-2} + p^{j-2} + \cdots + 1)$ and iterating we get the formula. Again the polynomiality is most easily verified by explicit computation. □

Remark 12.8 This time formula iii) can be rewritten

$$\frac{1}{2} \frac{p^{k-1} - 1}{p^m + 1} \left(\prod_{i=1}^{m-k+1} (p^i - 1) \right) \begin{bmatrix} m \\ k - 1 \end{bmatrix}_{p^2}.$$

Remark 12.9 It is not unreasonable to conjecture that the $\overline{\mathcal{V}}_{w_k}$ are complete in the (open) moduli space for $k \geq 3$. Moreover, the class λ_1 is conjectured to be ample on the moduli space. In characteristic 0, this follows from Baily–Borel. [Recently, this has been proved also in positive characteristic under mild conditions by Maulik [20] and Madapusi Pera [18,19].] If this is true then the open strata $\mathcal{V}_{w_k}$ for $k \geq 3$ are affine.

13 Applications

We shall now discuss two applications both pertaining to the even case.

13.1 (Quasi-)Elliptic fibrations with a section

If X is a K3 surface and $f : X \to \mathbb{P}^1$ is an elliptic (or possibly quasi-elliptic in characteristic 3) fibration with a section $E \subset X$, then E and a general fiber F span a hyperbolic plane $\mathbb{H}$ in $\mathrm{NS}(X)$ thus giving a $\mathbb{H}$-marking of X. Let now $\mathcal{M}^{es}$ be the stack of K3 surfaces together with a (quasi-)elliptic fibration (with base $\mathbb{P}^1$) with a chosen section on it. As the choice of an ample line bundle is not part of the choices made, let us take a moment to explain why this is an algebraic stack. We can consider the stack of K3-like surfaces (i.e., a surface with rational double points only as singularities and whose minimal resolution is a K3 surface) with an (quasi-)elliptic fibration with a section and irreducible fibers which is smooth along the section. Three times a fiber plus the section is an ample divisor, and hence, the stack of such surfaces is algebraic. Then, $\mathcal{M}^{es}$ is the Artin–Brieskorn simultaneous resolution stack of it.

Proposition 13.1 *$\mathcal{M}^{es}$ is of twisted even type of rank 20 so that Theorem 12.7 applies with $m = 10$. In particular $\sigma_0 = 10$ is not possible.*

Proof We start by verifying the versality hypothesis for $\mathcal{M}^{es}$. Let us therefore fix a geometric point $X \to \mathrm{Spec}(\mathbf{k})$. Recall that deformations of K3 surfaces are unobstructed, and the derivative of the period map $H^1(X, T_X) \to \mathrm{Hom}(H^0(X, \Omega^2_X), H^1(X, \Omega^1_X))$ is an isomorphism. Consider now the closed (formal) subscheme A of some formal universal deformation $\mathcal{X} \to S$ of X defined by the condition that the $\mathbb{H}$-marking of X extend over A. Then, A is defined by two equations and its tangent space, as a subspace of $H^1(X, T_X)$ is given by the condition that $v \cdot c_1(\mathcal{L}) = 0 \in H^2(X, \mathcal{O}_X)$ for all line bundles $\mathcal{L} \in \mathbb{H}$ and where $c_1(\mathcal{L}) \in H^1(X, \Omega^1_X)$ is the Hodge cohomology Chern class (induced by $\mathrm{dlog} : \mathcal{O}^*_X \to \Omega^1_X$). As the degree of the polarization is prime to p, the class c_1 gives an injection $\mathbb{H} \otimes \mathbf{k} \hookrightarrow H^1(X, \Omega^1_X)$ and hence the codimension of $T_s(A)$ in $T_s(S)$, where s is the closed point of S, is 2 and hence A is smooth. Furthermore, it also follows that $T_0(A)$ maps isomorphically onto $\mathrm{Hom}(H^0(X, \Omega^2_X), P)$, where P is the primitive part of $H^1(X, \Omega^1_X)$. This gives the required versality for the stack of $\mathbb{H}$-marked surfaces. What remains to show is that if the marking of X comes from a (quasi-)elliptic fibration with a section then so does any deformation of it. For the fibration, we let $\mathcal{L} = \mathcal{O}_X(F)$ be the line bundle of a fiber F. Then, $H^1(X, \mathcal{L}) = 0$, and hence, for any extension of it (to some closed subscheme of S), its direct image will be a vector bundle $\mathcal{E}$ of rank 2 which gives a map to the $\mathbb{P}(\mathcal{E})$-bundle extending the (quasi-)elliptic fibration. Similarly, the section is a (-2)-curve E and as $H^1(X, \mathcal{O}_X(E)) = 0$

any extension of $\mathcal{O}_X(E)$ will give an extension of the curve which then is a section. By construction, both line bundles extend over A.

As the discriminant of $\mathbb{H}$ is -1, we get by Theorem 7.1 that the Hodge discriminant of the primitive part is equal to 1. Then, from Proposition 5.3 (and the fact that in the notations of that proposition $m = 10$), we conclude that we are in the twisted case. Theorem 12.7 then gives the classes of the height and Artin invariant strata (together with the fact that $\sigma_0 = 10$ is not possible). □

Remark 13.2 There is an alternative way of excluding $\sigma_0 = 10$ similar to the way Artin excluded $\sigma_0 = 11$ for a general supersingular K3 surface. By [1] a supersingular (quasi-)elliptic K3 surface X has $\rho = 22$ and by the fact that $\mathbb{H}$ is unimodular we get that $\mathrm{NS}(X) = \mathbb{H} \perp P$. If $\sigma_0(X) = 10$, then the scalar product on P is divisible by p and $P(1/p)$ is a unimodular even negative definite form of rank 20 which is not possible as its index, 20, is not divisible by 8. This argument has the advantage of working also for $p = 2$.

Since the first version of this paper was written two alternative proofs of Proposition 13.1 have appeared, namely in the paper by Kondo and Shimada [16] and also in Liedtke's paper [17].

13.2 The canonical double cover of an Enriques surface

We let N be the lattice $E_{10}(-1) = \mathbb{H} \perp E_8(-1)$, and we fix a chamber (inside of the positive cone) with respect to the roots of N (see [8, II:§5] for a discussion of chambers in $E_{10}(-1)$). Let $\mathcal{M}^E$ be the moduli stack of marked Enriques surfaces where a marking is an isometry between the standard Enriques lattice $N_{10} = \mathbb{H} \perp E_8(-1)$ and the Néron–Severi group taking the fixed chamber into the ample cone of the Néron–Severi group. We can then construct $\mathcal{M}^{E,d} \to \mathcal{M}^E$, the moduli stack of canonical double covers of marked Enriques surfaces (i.e., while $\mathcal{M}^E(S)$, for a scheme S, is the groupoid of families of marked Enriques surfaces over S, $\mathcal{M}^{E,d}(S)$ is the groupoid of families of marked Enriques surfaces together with an unramified double cover of the Enriques surface which is fiberwise non-trivial).

Remark 13.3 Note that $\mathcal{M}^{E,d} \to \mathcal{M}^E$ is not an isomorphism but rather a (non-trivial) $\mathbb{Z}/2$-gerbe. The non-triviality is reflected in the fact that given a family $X \to S$ of Enriques surfaces a canonical double cover is a double cover $X' \to X$ which is non-trivial over every geometric fiber over S. There is an obstruction in $H^2(S, \mathbb{Z}/2)$ which in general is nonzero to the existence of such a cover, making "canonical double cover" something of a misnomer.

Pulling back the Néron–Severi group along the universal double cover $\mathcal{X}' \to \mathcal{X}$ over $\mathcal{M}^{E,d}$ we get a marking by $N(2)$ of the family $\mathcal{X}' \to \mathcal{M}^{E,d}$ of K3 surfaces.

Proposition 13.4 *$\mathcal{M}^{E,d}$ is of twisted even type of rank 12 so that Theorem 12.7 applies with $m = 6$. In particular $\sigma_0 = 6$ is not possible.*

Proof Again, we start by verifying that $\mathcal{M}^{E,d}$ fulfills the versality condition. We use the fact that $\mathcal{M}^{E,d}$ also can be described as the stack of K3 surfaces together with ι, a fixed point free involution. Its tangent space is then the space of linear maps $H^0(X, \Omega^1_X) \to H^1(X, \Omega_X)$ commuting with the involution. Now, ι acts by -1 on $H^0(X, \Omega^1_X)$ and by $+1$ on $N(2) \otimes \mathbf{k}$ and -1 on P under the decomposition $H^1(X, \Omega_X) = N(2) \otimes \mathbf{k} \perp P$ which gives what we want.

The marking has discriminant -2^{10} which is -1 up to squares just as in the previous example. Hence, we are in the twisted even case (with $m = 6$) and again Theorem 12.7 applies. □

Remark 13.5 Also, in this case, there is an arithmetic proof of the impossibility of $\sigma_0 = 6$ (using as above that the scalar product on P is divisible by p and that its rank is not divisible by 8). The proof does not extend to characteristic 2, however, as the polarization is not of degree prime to 2 (and the situation is in fact quite different in characteristic 2).

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