Divisors and the divisor class group.

X (compact) RS

Def $D_1 \sim D_2$ (linearly equivalent) if and only if $\exists$ merom. f^n f, $f \not\equiv 0$ with $\operatorname{div}(f) = D_1 - D_2$

equivalence relation

A divisor $\operatorname{div}(f)$ is called a <u>principal divisor.</u>

$\operatorname{Div}(X) / \operatorname{Prin}(X)$ = divisors on X / principal divisors

=: divisor class gp.

Given a divisor D we constructed a sheaf $\mathcal{O}_X(D)$

it is a locally free sheaf of rk 1 of $\mathcal{O}_X$-modules) (i.e. $\forall\, U$ $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ via the module structure and $\mathcal{F}(U) \cong \mathcal{O}_X(U)$ as $\mathcal{O}_X(U)$-modules for sufficiently small U.

$$D \longrightarrow \mathcal{O}_X(D)$$

If $D_1 \sim D_2$ then $\mathcal{O}_X(D_1) \xrightarrow{\sim} \mathcal{O}_X(D_2)$

$D_1 - D_2 = (f)$ $\qquad \varphi \qquad \varphi \cdot f$

If $\mathcal{U} = \{U_i : i \in I\}$ is a cover so that $D|U_i$ is the divisor of f_i, a meromorphic function on U_i then a section $s \in \Gamma(X, \mathcal{O}_X(D))$ is locally given by $s_i \in \mathcal{O}_X(U_i)$ s.t $s_j = f_j/f_i \cdot s_i$

$i \in I$

We can associate to D a 1-cocycle as follows

$\mathcal{U} = \{U_i : i \in I\}$ open cover such that $D|_{U_i} = \mathrm{div}(f_i)$

f_i merom. on U_i

Then set
$$f_{ij} = f_j / f_i \quad \text{on } U_i \cap U_j$$
$$\in \mathcal{O}_X^*(U_i \cap U_j)$$

gives cochain in $C^1(\mathcal{U}, \mathcal{O}_X^*)$

We have $\quad f_{ik} = f_k/f_i = f_j/f_i \cdot f_k/f_j = f_{ij} \cdot f_{jk}$

$$f_{ii} = 1$$

so we have a 1-cocycle $Z^1(\mathcal{U}, \mathcal{O}_X^*)$.

hence an elt in $H^1(\mathcal{U}, \mathcal{O}_X^*)$, i.e. in $H^1(X, \mathcal{O}_X^*)$

If $D = \mathrm{div}(g)$, then $\mathrm{div}(f_i) = \mathrm{div}(g)$ on U_i

hence $\quad g_i = f_i / g \in \mathcal{O}_X^*(U_i)$

and
$$f_{ij} = g_j / g_i, \quad \text{hence a co-boundary.}$$

So we get

$$\mathrm{Div}(X)/P(X) \hookrightarrow H^1(X, \mathcal{O}_X^*). \qquad \underline{\text{Picard group}}$$

divisor class gp.

But this map is also surjective: take a cocycle

$\{f_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$; take U_{i_0} ~~small~~ ~~[illegible]~~ fixed

~~[illegible]~~

take $f_i = \begin{cases} 1 & \text{on } U_{i_0} \\ f_{i_0 i} & \text{on } U_i \end{cases}$

$\mathrm{div}(g_i) \qquad g_i = f_{ii_0}$

Then $\mathrm{div}(f_{ii_0})$ on U_i defines locally a divisor

but since on $U_i \cap U_j$ the function $f_{ii_0}/f_{ii_0} =$ is invertible

this divisor is well defined.

We thus find

Prop $\mathrm{Div}(X)/P(X) \cong H^1(X, \mathcal{O}_X^*)$

Remark: This group can also be identified with the group of $\cong$ classes of locally free $\mathcal{O}_X$-modules of rk 1

(group operation: $\cdot \otimes_{\mathcal{O}_X} \cdot$)

X RS., say compact

We have a ~~long~~ short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

this gives a long exact sequence

$$0 \to \overset{\mathbb{Z}}{H^0(X, \mathbb{Z})} \to \overset{\mathbb{C}}{H^0(X, \mathcal{O}_X)} \to \overset{\mathbb{C}^*}{H^0(X, \mathcal{O}_X^*)}$$

$$\xrightarrow{?} H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*)$$

$$\to H^2(X, \mathbb{Z}) \to 0$$

first line: $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^*$ is exact

hence we get

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to 0$$

We want to analyze $H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z})$

Note that $H^2(X,\mathbb{Z}) \xrightarrow[\text{natural map}]{\supset} H^2(X,\mathbb{C})$ is injective

(Pf. If $\{f_{ijk}\} \in Z^2(\mathcal{U},\mathbb{Z})$ becomes a coboundary in $Z^2(\mathcal{U},\mathbb{C})$, then exist $\varphi_{ij} \in C^1(\mathcal{U},\mathbb{C})$ s.t.

$$f_{ijk} = \varphi_{ij} - \varphi_{ik} + \varphi_{jk}$$

but φ_{ij} are not necessarily ~~integral~~ valued; but $\psi_{ij} = \exp(2\pi\sqrt{-1}\,\varphi_{ij})$ satisfies

$$\psi_{ik} = \psi_{ij}\psi_{jk}$$

hence put $g_{ij} = \varphi_{ij} - \frac{1}{2\pi i}\log\psi_{i0} + \frac{1}{2\pi i}\log\psi_{0j}$ $\mathbb{Z}$-valued

and $g_{ij} - g_{ik} + g_{jk} = f_{ijk}$.)

$$H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C}) \cong H^1(X,\Omega^1_X) \cong \mathbb{C}$$

from $0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$

$\{f_{ij}\} \in Z^1(X,\mathcal{O}_X^*)$

$\delta(f_{ij})$: $\log f_{ij} - \log f_{ik} + \log f_{jk}$

$\Rightarrow$ $d\log f_{ij}$ in $Z^1(\mathcal{U},\Omega^1_X)$

$$\mathrm{Res}(d\log f_{ij}) = \sum \#\text{ zeros} - \sum \#\text{ poles}$$

$$= \deg(D)$$

hence $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$

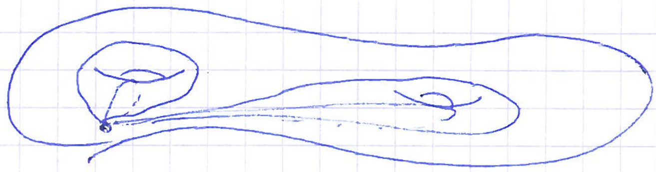
$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \qquad \text{exponential sequence}$$

$\rightsquigarrow$

$$\begin{array}{ccccccc} 0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to & H^1(X,\mathcal{O}_X^*) & \to & H^2(X,\mathbb{Z}) \\ & \| & & \wr\| \\ & \mathrm{Pic}(X) & \longrightarrow & \mathbb{Z} \\ & [D] & \longrightarrow & \deg D \end{array}$$

<u>Conclusion</u> $\quad \mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/P(X) \simeq H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$

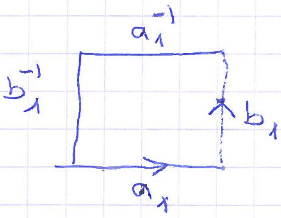
Now use $\pi_1(X)$



$a_1, b_1, a_2, b_2, \ldots, a_g, b_g$

cut open.

$g = 1$



$$a_1 b_1 a_1^{-1} b_1^{-1} = 1$$

more generally: π_1 generated by $a_1, b_1, \ldots, a_g, b_g$ with one relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

Abelianized $\pi_1 \simeq \mathbb{Z}^{2g}$ generated by $a_1, b_1, \ldots, a_g, b_g$

hence $H_1(X,\mathbb{Z}) = \mathbb{Z}^{2g}$

hence $H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$, but this will also follow later

$0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{d} \Omega^1 \to 0$ exact induces

$$\begin{array}{ccccccccc} 0 \to H^0(X, \Omega^1_X) & \xrightarrow{\delta} & H^1(X, \mathbb{C}) & \to & H^1(X, \mathcal{O}_X) & \to & H^2(X, \mathbb{C}) & \to 0 \\ & & \uparrow & & & & \| \wr & \\ & & H^1(X, \mathbb{R}) & & & & \mathbb{C} & \end{array}$$

Claim: If $\delta\omega$ (with $\omega \in H^0(X, \Omega^1_X)$) lies in (the image of) $H^1(X, \mathbb{R})$ then $\omega = 0$

Remark: $H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{R}) \hookrightarrow H^1(X, \mathbb{C})$

same proof as above.

Pf of claim: Suppose $\delta\omega$ lies in $H^1(X, \mathbb{R})$

Then $\exists$ open cover $\mathcal{U} = \{U_i : i \in I\}$ of X and

$f_i \in \mathcal{O}_X(U_i)$ st. $\omega = df_i$ on U_i

$f_j - f_i$ real on $U_i \cap U_j$

Put $g_j = e^{2\pi\sqrt{-1} f_j}$ holomorphic on U_i

$|g_i| = |g_j|$ on $U_i \cap U_j$

(so this defines a globally defined function

it attains a maximum

hence g_j attains a local maximal for some j

hence this g_j constant

hence g_i constant

$\Rightarrow df_j = 0 \Rightarrow \omega = 0$.

$H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{C})$ as we saw.

(

finitely generated

say $\eta_1, \dots, \eta_m$ generators

Suppose $\sum_{i=1}^{m} \lambda_i \eta_i = 0$ in $H^1(X, \mathcal{O}_X)$

$\lambda_i \in \mathbb{R}$

then $\sum \lambda_i \eta_i \in \delta H^0(X, \Omega^1_X) \cap H^1(X, \mathbb{R})$

hence zero.

So we get an embedding $H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O}_X) = \mathbb{C}^g$

and the images of η_i ~~span~~ are $\mathbb{R}$-~~basis~~ linearly indep.

$\Rightarrow$ $m \leq 2g$. We shall later see $m = 2g$.

[Define $\pi_1 \ni \gamma \longmapsto (\omega \mapsto \int_\gamma \omega)$

linear map on $H^0(X, \Omega^1_X)$

gives a map use $H^0(X, \Omega^1_X)^\vee = H^1(X, \mathcal{O}_X)$

$$\pi_1 \longrightarrow \pi_1^{ab} \longrightarrow H^1(X, \mathcal{O}_X)$$

$$\pi_1^{ab} = \mathbb{Z}^{2g}$$

~~$\int_\gamma \omega$~~

Now use $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\pi_1(X), \mathbb{Z})$

or make explicitly a real 1-form τ such that $\int_{[\gamma]} \tau \neq 0$]

~~$\mathbb{Z}^{2g} \hookrightarrow H^1(X,$~~

$$\mathrm{Jac}(X) = H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z})$$

$H^1(X, \mathbb{Z})$ free $\mathbb{Z}$-module of rk $m \leq 2g$

span $\mathbb{R}$-linearly a lattice $\mathbb{Z}^m$

$\hookrightarrow$ $\mathrm{Jac}(X) \cong$ complex torus $\times$ complex vector space

We will see in a moment that $m = 2g$.

D divisor $\rightsquigarrow$ $H^0(X, \mathcal{O}_X(D))$ finite dim vector space

$\downarrow$

$\mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ a projective space

$$\mathbb{P}(H^0(X, \mathcal{O}_X(D))) \longleftrightarrow \underbrace{\{D' : D' \geq 0,\ D' \sim D\}}_{= |D|}$$

$$\mathbb{C} \cdot f \longleftrightarrow \mathrm{div}(f) + D = D' \geq 0$$

Make now map

$$X^g \xrightarrow{\alpha} H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z})$$

Q base pt

$$(P_1, \dots, P_g) \to \left[\sum_{i=1}^{g} (P_i - Q) \right]$$

We looked at the exact sequence of sheaves (X cpt RS)

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

and get

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \underset{\substack{\| \\ \mathrm{Pic}(X)}}{H^1(X,\mathcal{O}_X^*)} \xrightarrow{\deg D} \underset{\substack{\cong \\ \mathbb{Z}}}{H^2(X,\mathbb{Z})} \to 0$$

$$\mathrm{Ker}(\deg) =: \mathrm{Pic}^0(X) \cong H^1(X,\mathcal{O}_X) / H^1(X,\mathbb{Z})$$

We know $H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathbb{C})$

injective, so torsion free ($\cong \mathbb{Z}^m$)

image vectors of basis are $\mathbb{R}$-linearly independent.

($\Rightarrow m \leq 2g$)

(using
$$0 \to H^0(X,\Omega^1) \to \underset{\substack{\uparrow \\ H^1(X,\mathbb{R})}}{H^1(X,\mathbb{C})} \to H^1(X,\mathcal{O}_X) \to \underset{\substack{\cong \\ \mathbb{C}}}{H^2(X,\mathbb{C})}$$
)

We define $\mathrm{Jac}(X) = \mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_X) / H^1(X,\mathbb{Z})$

We made a map

$$X^g \xrightarrow{\alpha} H^1(X,\mathcal{O}_X) / H^1(X,\mathbb{Z})$$

Choose a base point $Q \in X$

$$(P_1,\dots,P_g) \to \Big(\sum_{i=1}^{g} (P_i - Q)\Big)$$

divisor class

<u>Claim</u> : α surjective

Pf D divisor (class) of degree 0. Then $D + gQ$ is of degree g ; $h^0(D+gQ) \geq 1$ by RR.

hence $\underbrace{(f) + D + gQ}_{= D'} \geq 0$

$D' \sim D$, $D' \geq 0$ linearly equivalent

Q.e.d.

hence $X^g \twoheadrightarrow \mathrm{Pic}^0(X)$ using base pt Q

or without using a base pt

$X^g \twoheadrightarrow \mathrm{Pic}^g(X)$ — divisor classes of degree g

X^g compact, continuous map, hence $\mathrm{Pic}^{(0)}_{g}(X)$ is compact, hence $\mathrm{rk}\, H^1(X, \mathbb{Z}) \geq 2g$ $\quad (\dim_{\mathbb{R}} H^1(X, \mathcal{O}_X) = g)$

but we saw $m \leq 2g$, so

<u>Cor</u> $\underline{\mathrm{rk}\, H^1(X,\mathbb{Z}) = 2g}$, $\quad H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

We look at

$$\alpha_n : X^{(n)} = X^n / S_n \longrightarrow \mathrm{Jac}^{(n)}(X) = \mathrm{Pic}^{(n)}(X)$$

S_n symm gp

$\mathrm{Pic}^{(n)}(X)$ div. classes of deg n

surjective for $n \geq g$ by RR.

<u>Special case</u> $g \geq 1$

$$n = g-1 \qquad X^{(g-1)} \longrightarrow \mathrm{Jac}^{(g-1)}(X)$$
$$(P_1, \dots, P_{g-1}) \longrightarrow \left[\sum_{i=1}^{g-1} P_i\right]$$

image is the so-called <u>theta divisor</u> $\Theta \subset \mathrm{Jac}^{(g-1)}(X)$

By RR and Serre duality

$h^0(D) = h^0(K-D)$ for D of degree $g-1$
K canonical divisor

So we have an involution

$$[D] \longleftrightarrow [K-D] \quad \text{of } \begin{cases} \mathrm{Pic}^{g-1}(X) \\ \Theta . \end{cases}$$

fibres of α_n

$$\alpha_n(D_1) = \alpha_n(D_2) \Longleftrightarrow D_1 - D_2 = \underbrace{(f)}_{\text{divisor}}$$

so fibre of α_n over $[D]$ is

$$\{ D' \; . \; D' \geq 0, \deg D' = n, \; D' \underset{\text{lin equiv}}{\sim} D \} \qquad D + (f)$$
$$\| \qquad\qquad\qquad\qquad \uparrow$$
$$\mathbb{P}(H^0(X, \mathcal{O}_X(D))) \qquad\qquad \mathbb{C}^* \cdot f$$
$$\wr$$
projective space of dim $h^0(X, \mathcal{O}_X(D)) - 1$

<u>Remark</u> Image α_{g-1} is a divisor; if not then $X^{(g-1)}$ is close to a $\mathbb{P}^m$-fibration over its image; then it carries no holom. $(g-1)$-forms; but it does for $g \geq 1$.

<u>$g=0$</u> X of genus 0. ; P a pt of X

$$h^0(P) = \underset{\substack{\| \\ 0}}{h^0(K-P)} + 1 + 1 - g = 2$$

because $\underset{\substack{\| \\ -3}}{\deg(K-P)} < 0$

So $H^0(X, \mathcal{O}_X(P))$ contains besides the constant functions a non-constant function f with a simple pole in P

$f: X \xrightarrow{\pi} \mathbb{P}^1$, $\deg \pi = 1$, because $\pi^*(\infty) = P$

<u>So $X \cong \mathbb{P}^1$.</u>

<u>$g=1$.</u> $X \longrightarrow \mathrm{Jac}^{(1)}(X)$

$\{P\} \longrightarrow [P]$

this is an isomorphism. (fibres of α_n are always projective space; here only 1 pt. ; or use RR

if $|P|$ has $\dim \geq 1$, then $h^0(X, \mathcal{O}_X(P)) \geq 2$, hence $\exists$ function with one pole at $P \Rightarrow X \cong \mathbb{P}^1$, contradicts $g \neq 0$)

So $X \cong \mathrm{Jac}^{(1)}(X)$.

After choosing a base pt $Q \in X$ we get

$$X \cong \mathrm{Jac}^0(X) \quad \text{a group variety.}$$

So

$$X \cong \mathbb{C}/\Lambda = H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z})$$

$g=2$ $\quad \dim H^0(X, \Omega^1_X) = 2 \quad (= \dim H^1(X, \mathcal{O}_X))$

Define a map

$$X \xrightarrow{\varphi} \mathbb{P}^1 \qquad x \longrightarrow (w_1(x) : w_2(x))$$

with w_1, w_2 basis. (Well-defined; locally Ω^1_X trivial $U \times \mathbb{C}$; on $U_i \cap U_j$ we get $(f_{ij} w_1 : f_{ij} w_2)$ same ratio.

φ cannot ~~be~~ an isomorphism.

$$\varphi^{-1}(P) = \{ x \in X : w_2(x) = 0 \}$$

$P \in \mathbb{P}^1$, say $P = (1;0)$

$= (\alpha : \beta)$

$$\beta\, w_1(x) + \alpha\, w_2(x) = 0$$

hence $\deg \varphi^{-1}(P)) = 2g-2 = 2$

so the map

$X \xrightarrow{\varphi} \mathbb{P}^1$ is of degree 2

We can write $\mathbb{C}(X)$ as $\mathbb{C}(x,y)$

with $y^2 = f(x)$

Hurwitz - Zeuthen formula:

$$2g(X) - 2) = 2\,(2g(\mathbb{P}^1) - 2) + b$$

$$2 = -4 + b \quad \Rightarrow \quad b = 6$$

So 6 ramification pts

So we can write $y^2 = f(x)$ $\quad \deg f = 6$ or 5 (one ~~root~~ branch at ∞)

α_{g-1}.

$$X \hookrightarrow Jac^{(1)}(X) \xrightarrow{\sim} Jac^{0}(X)$$

$$P \to [P] \longmapsto [P-Q] \quad \text{base pt.}$$

(embedding because $P \sim P'$ means $g(X) = 0$)

So $\Theta \hookrightarrow Jac^{(1)}(X)$

$$X^{(2)} \longrightarrow Jac^{(2)}(X)$$

$$(P_1, P_2) \longmapsto [P_1 + P_2]$$

fibres: $P_1 + P_2 \sim R_1 + R_2$ with $P_1 + P_2 \neq R_1 + R_2 \in X^{(2)}$

then

$$h^0(P_1 + P_2) \geq 2.$$

RR: $\underset{\underset{2}{\vee}}{h^0(D)} - h^0(K-D) = 2+1-2 = 1$

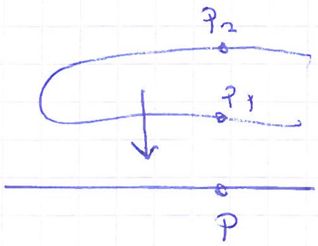
$$\Rightarrow \underbrace{h^0(K-D) \geq 0}_{\deg K-D = 0}$$

$\Rightarrow D = K$

So we see

$$Jac^{(2)}(X) = X^{(2)} \text{ with } \underbrace{\text{one } \mathbb{P}^1}_{\text{exceptional curve}} \text{ contracted}$$

blown down



P varies over $\mathbb{P}^1$ $\rightsquigarrow$ fibre!

We have 6 ramification points $P_1, \ldots, P_6$ on X.

$2P_i \sim 2P_j$ because images Q_i, Q_j on $\mathbb{P}^1$

$$\varphi^* Q_i = 2P_i \qquad Q_i \sim Q_j$$

$$\Rightarrow \quad 2P_i \sim 2P_j$$

$\Rightarrow 2(P_i - P_j) = 0 \Rightarrow P_i - P_j$ give $15 = \binom{6}{2}$ pts of order 2

$$\mathrm{Jac}^0(X) \cong \mathbb{C}^2/\Lambda$$

$\Lambda/2\Lambda$ 16 pts (15 + origin).

Recall: X RS, compact, genus g

$\mathbb{C}(X)$ function field; field of meromorphic functions

$X \to \mathbb{P}^1$ there is such a map of degree $\leq g+1$

(RR $h^0(D) - h^0(K-D) = \deg D + 1 - g \geq 2$)

So this implies that $\mathbb{C}(X)$ is an extension field of degree $\leq g+1$ of $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(x)$

Theorem of the primitive element says that $\mathbb{C}(X)$ is generated of over $\mathbb{C}(x)$ by one elt y

$\Rightarrow$ We get an equation

$$f(x,y).$$

So X is given as an <u>algebraic curve</u>. (could be singular)

<u>Canonical map</u> X cpt RS, genus $g \geq 2$

choose basis $\omega_1, .., \omega_g$ of $H^0(X, \Omega^1_X)$.
Make a map

$$X \xrightarrow{\varphi} \mathbb{P}^{g-1} \qquad \omega \longmapsto (\omega_1(x) : \quad \ldots \quad : \omega_g(x))$$

<u>Remark</u> For every pt $P \in X$ there exist $w \in H^0(X, \Omega^1_X)$ s.t. $w(P) \neq 0$.

<u>Pf</u>. If not, then $\underset{g}{\underset{\|}{h^0(K)}} = h^0(K-P)$ for this P

then $h^0(P) \underset{RR}{=} h^0(K-P) + 1 + 1 - g = 2$.

hence $\exists$ non-constant f giving $X \simeq \mathbb{P}^1$
but $g \neq 0$)

So φ is a well-defined holom. map

<u>$g=2$</u> $X \xrightarrow{2:1} \mathbb{P}^1$ not an embedding

<u>$g=3$</u> We have a map $X \longrightarrow \mathbb{P}^2$

Image not contained in a line (otherwise we have $\omega_3(P) = \alpha\, \omega_1(P) + \beta\, \omega_2(P)$ say for all P
but $\omega_1, \omega_2, \omega_3$ are linearly independent)

Take a line $\alpha x + \beta y + \gamma z = 0$ in $\mathbb{P}^2$

$$L \cap \varphi(X) = \{ x \in X : \alpha\, \omega_1(x) + \beta\, \omega_2(x) + \gamma\, \omega_3(x) = 0 \}$$

divisor of a holom 1-form
has degree 4 in our case

Possibilities: $\varphi(X)$ is a curve of degree 4

$\varphi(X)$ is a conic and $\deg\varphi = 2$

We see

$$\varphi : X \xrightarrow{\sim} \varphi(X) \text{ of degree } 4 \quad (\cong)$$

quartic curve

or $\varphi : X \xrightarrow{2:1} \varphi(X)$ conic $\cong \mathbb{P}^1$

$\deg \varphi = 2$

Indeed, if $\varphi(P_1) = \varphi(P_2)$ then $h^0(K-P_1) = h^0(K-P_1-P_2)$
then by RR

$$h^0(K-P_1-P_2) = h^0(P_1+P_2) + (2g-4) + 1-g$$

$$\begin{aligned}&\| \\ &h^0(K-P_1) \\ &\| \\ &g-1\end{aligned}$$

, hence $h^0(P_1+P_2) = 2$

ie $\exists$ non-constant f with $(f) + P_1+P_2 \geq 0$

f defines a map $C \xrightarrow{2:1} \mathbb{P}^1$ of degree 2

X

<u>Def</u> A RS of genus ≥ 2 is called hyperelliptic if there exists a non constant holom map

$$X \to \mathbb{P}^1 \text{ of degree } 2$$

(Equivalently, $\exists$ P_1, P_2 such that $|P_1+P_2|$ has projective dimension ≥ 1).

Look at α_{g-1} for this case

$$X^{(2)} \longrightarrow \mathrm{Pic}^{(2)}(X)$$

image is theta divisor $\Theta \subset \mathrm{Pic}^{(2)}(X)$

fibre: if (P_1+P_2) has a pos. dim. fibre then

$|P_1+P_2|$ has dim ≥ 1, i.e. X is hyperelliptic.

~~so~~ then $X^{(2)} \longrightarrow \mathrm{Pic}^{(2)}(X)$ contracts a $\mathbb{P}^1$

X hyperelliptic

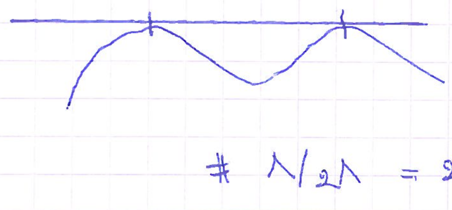
or $X^{(2)} \longrightarrow \mathrm{Pic}^{(2)}(X)$ is an embedding

and the theta divisor is a smooth surface

(X plane quartic)

Quartic curves have a beautiful geometry

28 bitangents



$L \cap \varphi(X) = 2P_1 + 2P_2$

$\# \Lambda/2\Lambda = 2^6 = 64 = 28 + 36$

$2^{3-1}(2^3-1)$

hence $2P_1 + 2P_2 \sim K$

hence $2(P_1+P_2)$ effective canonical divisor.

"points of order 2 on Θ"

We have 28 such ~~pairs~~ bitangents.

24 inflexion pts