Solutions Test Riemann Surfaces

—

1. If $f : X \to \mathbb{C}$ is a holomorphic map then consider the holomorphic function $g = e^{-2\pi i f}$. Apply the maximum principle to g.

2.

i) In order to find for a general point $w \in \mathbb{C}$ the fibre $\pi^{-1}(w)$ we must solve the equation $z^2 + 1/z^2 = w$, that is $z^4 - wz^2 + 1 = 0$. Since this is a degree 4 equation we find for general w four solutions, hence the degree of π is 4.

ii) We begin by remarking that $z \mapsto 1/z$ and $z \to -z$ are automorphisms of $\mathbb{P}^1$ which commute with π. So the group $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts on the fibres of π. Moreover by using $(z, w) \mapsto (iz, -w)$ we see that the ramification behavior in the fibre over w is the same as for $-w$.

The branch points over $w \in \mathbb{C} \subset \mathbb{P}^1$ occur if the equation $z^4 - wz^2 + 1 = 0$ has coinciding roots. Also $w = \infty$ can be a branch point.

Viewing the equation as a quadratic equation in z^2 the case of coinciding roots happens if the discriminant $w^2 - 4$ vanishes, that is for $w = \pm 2$. So these are the finite branch points. For $w = \infty$ we must have $z = 0$ or $z = \infty$ and thus it is a branch point.

iii) We first deal with the ramification index near $z = 0$. Change coordinates by putting $w = 1/u$. Then we get $uz^4 + u - z^2 = 0$, that is, $u(z^4 + 1) = z^2$. Since $z^4 + 1$ is invertible near $z = 0$ it is clear that the ramification index is 2. (We can find a holomorphic function h near $z = 0$ such that $h^2 = 1 + z^4$ and then we replace z by z/h such that the local form of the map becomes $z \mapsto u = z^2$.)

For the points over $w = 2$ we have the equations $z^2 = (w \pm \sqrt{w^2 - 4})/2$ from which it follows that $z = \pm 1$ are the solutions. This clearly shows that the ramification index is 2 for $z = 1$ and $z = -1$ over $w = 2$. So the fibre type over all branch points is $4 = 2 + 2$. (This checks with the genus in the Hurwitz-Zeuthen formula: $-2 = 4 \cdot (-2) + 1 + 1 + 1 + 1 + 1 + 1$.)

3. Let $\pi : X \to Y$ be a holomorphic map of compact Riemann surfaces of degree d. Then $2g(X) - 2 = d(2g(Y) - 2) + b$, where $b = \sum_{P \in X}(r_P - 1)$ with r_P the ramification order at P.

Proof. Let P be a point of X with image $Q = \pi(P)$. Then we can find local coordinates at P and Q, say z and w, such that π locally at P is given by $z \mapsto z^r = w$. If ω is a meromorphic differential form on Y such that $\omega = f(w)dw$ near Q then $\pi^*(\omega)$ is of the form $f(z^r)d(z^r) = r\, f(z^r)z^{r-1}dz$ near P and hence

$$\mathrm{ord}_P(\pi^*(\omega)) = r\mathrm{ord}_Q(f) + (r - 1) = r\mathrm{ord}_Q(\omega) + (r - 1)$$

We know that if $P_1, \ldots, P_t$ are the points of the fibre then $\pi^*(Q) = r_1 P_1 + \ldots + r_t P_t$, with $r_1 + \cdots + r_t = d$, hence

$$\begin{aligned}\deg(\operatorname{div}(\pi^*\omega)) &= \sum_Q \Big(\sum_{P:\pi(P)=Q} r_P \operatorname{ord}_Q(\omega) + (r_P - 1)\Big) \\ &= d \sum_Q \operatorname{ord}_Q \omega + \sum_{P \in X} (r_P - 1) \\ &= d(2g(Y) - 2) + b\end{aligned}$$

4.

i) To show that the short exact sequence of sheaves is exact we need to prove exactness at the stalks. For any open U we have an embedding $\mathbb{C}(U) \subset \mathcal{E}(U)$. This implies that we have an injective map $\mathbb{C}_x \to \mathcal{E}_x$ on the stalks. The kernel of d on $\mathcal{E}(U)$ consists of the constant functions. Since $d \cdot d = 0$ it follows that $d\mathcal{E}(U)$ consists of closed forms. Furthermore, let $x \in X$. Near x every closed 1-form ζ can locally be written as df for some function $f \in \mathcal{E}(V)$ on a sufficiently small neighborhood of x. This shows that d is surjective on the stalk $\mathcal{Z}(U)$.

ii) We now form the long exact cohomology sequence:

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{Z}) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{E}) = 0$$

where the last zero follows from the fact that $\mathcal{E}$ is fine. We can thus rewrite this sequence as

$$0 \to \mathbb{C} \to \mathcal{E}(X) \xrightarrow{d} \mathcal{Z}(X) \to H^1(X, \mathbb{C}) \to 0.$$

In other words we get the Theorem of de Rham:

$$H^1(X, \mathbb{C}) \cong \mathcal{Z}(X)/d\mathcal{E}(X) = \text{closed 1-forms modulo exact forms}$$

5.

i) By Serre duality we have $H^1(X, O_X(D))^\vee \cong H^0(X, O_X(K-D))$ with K a canonical divisor. The degree of $K - D$ is negative, hence $H^0(X, O_X(K - D)) = (0)$. We used Serre duality

$$H^0(X, \Omega^1_X(-D)) = H^1(X, O_X(D))^\vee$$

and the fact that $H^0(X, O_X(D)) = (0)$ if $\deg(D) < 0$. We also observe that $\Omega^1_X(E) \cong O_X(K + E)$ for any divisor E and canonical divisor K.

ii) We have $H^1(X, O_X(D - P)) = (0)$ by i) since $\deg(D - P) > 2g - 2$. Hence we get from the short exact sequence $0 \to O_X(D - P) \to O_X(D) \to \mathcal{F} \to 0$ with $\mathcal{F} = \mathbb{C}_P$ the skyscraper sheaf at P, the exact sequence

$$0 \to H^0(X, O_X(D - P)) \to H^0(X, O_X(D)) \to \mathbb{C} \to 0$$

and this shows that the map is not surjective.

iii) If $\operatorname{ord}_P(D) = n_P$ then there exists an element $f \in H^0(X, O_X(D))$ which does not lie in $H^0(X, O_X(D-P))$, and that means that $\operatorname{ord}_P(f) = -n_P$. Then $E = \operatorname{div}(f) + D$ is effective, linearly equivalent to D and has $\operatorname{ord}_P(E) = 0$.

6.

i) The space $H^0(X, O_X(P))$ contains the constant functions. Suppose that the space $H^0(X, O_X(P))$ contains a non-constant function f. Then f defines a holomorphic map $f : X \to \mathbb{P}^1$ which is of degree 1 as f has 1 pole. Hence f is an isomorphism; this contradicts the fact that $g \neq 0$.

ii) By Riemann-Roch we have

$$h^0(K-P) = h^1(K-P) + 1 + \deg(K-P) - g = h^1(K-P) + g - 2.$$

But by Serre duality $h^1(K-P) = h^0(P)$ and this equals 1 as observed in i).

iii) By Riemann-Roch and Serre duality we have

$$\begin{aligned} h^0(K-P-Q) =& h^0(P+Q) + (2g-4) + 1 - g = h^0(P+Q) + g - 3 \\ h^0(K-P) =& h^0(P) + (2g-3) + 1 - g = g - 1 \end{aligned}$$

and we thus have $h^0(P+Q) = 2$.

iv) Since $\dim H^0(X, \Omega^1_X) = g$ and $\dim H^0(X, \Omega^1_X(-P)) = h^0(K-P) = g-1$ there exists a holomorphic 1-form that does not vanish at a given point P. Hence not all elements ω_i vanish at P. So the map π defined by $x \mapsto (\omega_1(x) : \ldots : \omega_g(x))$ is well-defined.

v) If the map π is not injective then there exist distinct points P and Q such that if ω vanishes at P it also vanishes at Q. But this means that $H^0(X, \Omega^1_X(-P-Q)) = H^0(X, \Omega^1(-P))$. The result follows from iii).