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Rank one Eisenstein cohomology of local systems on the moduli space of abelian varieties

To Fabrizio Catanese on the Occasion of his 60th Birthday

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Abstract We give a formula for the Eisenstein cohomology of local systems on the partial compactification of the moduli of principally polarized abelian varieties given by rank 1 degenerations.

Keywords Eisenstein cohomology, BGG complex

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1 Introduction

The cohomology of local systems on Shimura varieties is a rich source of information about modular forms. In order to distill this information from the cohomology one must separate the contribution of the boundary of the Shimura variety to the cohomology from the contribution of the interior. The contribution from the boundary is called Eisenstein cohomology. In this paper we give a recursive formula for the main part of the Eisenstein cohomology of a local system on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. By the main part we mean the contribution from the largest boundary component of the Satake compactification of $\mathcal{A}_g$, the boundary component that parametrizes semi-abelian varieties of torus rank one.

The study of Eisenstein cohomology was initiated by Harder [12, 13] and carried on by his students Schwermer and Pink [14, 17], cf. also [10]. They used the Borel-Serre compactification and topological methods to study it. Here we use coherent cohomology in the form of the BGG-complex introduced by Faltings [8, 9].

Besides providing a recursive formula for the rank one part of the Eisenstein cohomology for general g we prove an explicit formula for the full Eisenstein cohomology in the case $g = 2$. This formula was conjectured in [7] and was essential in deriving the traces of the Hecke operators on spaces of vector-valued Siegel modular forms from our results on counting points on curves of genus 2 in [7] and [1].

The recursive formula for the rank one part of the Eisenstein cohomology played an important role in guessing a conjectural formula for the traces of the Hecke operators on spaces of Siegel modular forms of genus 3 in [2]. Surprisingly, the form of the Eisenstein cohomology there seems to follow the pattern of the rank one part.

2 Definitions and results

Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian varieties over a field k. The universal abelian scheme $\pi : \mathcal{X}_g \to \mathcal{A}_g$ over $\mathcal{A}_g$ defines a local system $\mathbb{V} = R^1\pi_*\mathbb{Q}$ of rank $2g$ on $\mathcal{A}_g$, and for a prime ℓ different from the characteristic of k its ℓ-adic variant $\mathbb{V} = R^1\pi_*\mathbb{Q}_\ell$ for the étale topology. This local system $\mathbb{V}(1)$ is associated with the standard representation of $\mathrm{GSp}(2g,\mathbb{Q})$. To an irreducible representation of $\mathrm{Sp}(2g,\mathbb{Q})$ with the highest weight $\lambda = (\lambda_1,\dots,\lambda_g)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_g$ we can associate a local system $\mathbb{V}_\lambda$ of weight $\sum_i \lambda_i$ (this determines the lift to $\mathrm{GSp}(2g,\mathbb{Q})$) which occurs 'for the first time' in

$$\bigotimes_{j=1}^{g-1} \mathrm{Sym}^{\lambda_j-\lambda_{j+1}}(\wedge^j\mathbb{V}) \otimes \wedge^g\mathbb{V}^{\lambda_g}.$$

The cohomology of this local system is closely related to vector-valued Siegel modular forms, cf. [5, 7, 9]. We are interested in the Euler characteristic

$$e(\mathcal{A}_g,\mathbb{V}_\lambda) := \sum(-1)^i\,[H^i(\mathcal{A}_g,\mathbb{V}_\lambda)]$$

in the Grothendieck group of an appropriate category, i.e., mixed Hodge structures for complex cohomology or of Galois representations if one works with étale cohomology. Similarly, we can consider the analogue for compactly supported cohomology

$$e_c(\mathcal{A}_g,\mathbb{V}_\lambda) := \sum(-1)^i\,[H^i_c(\mathcal{A}_g,\mathbb{V}_\lambda)].$$

We note that this cohomology is zero if $\sum\lambda_i$ is odd.

There is a natural map $H^*_c(\mathcal{A}_g,\mathbb{V}_\lambda) \to H^*(\mathcal{A}_g,\mathbb{V}_\lambda)$ and the image is called the interior cohomology. We define the Eisenstein cohomology as the difference

$$e_{\mathrm{Eis}}(\mathcal{A}_g,\mathbb{V}_\lambda) := e(\mathcal{A}_g,\mathbb{V}_\lambda) - e_c(\mathcal{A}_g,\mathbb{V}_\lambda).$$

Instead of looking at a full compactification of $\mathcal{A}_g$ we consider the partial compactification of rank $\leqslant 1$ degenerations. One can obtain it by considering a compactification $\tilde{\mathcal{A}}_g$ of Faltings-Chai-type together with the natural map $q : \tilde{\mathcal{A}}_g \to \mathcal{A}^*_g$ to the Satake compactification. The Satake compactification has a stratification $\mathcal{A}^*_g = \bigsqcup_{i=0}^g \mathcal{A}_i$. Then the partial compactification $\mathcal{A}'_g$ is defined as $q^{-1}(\mathcal{A}_g \sqcup \mathcal{A}_{g-1})$ and can be viewed as the moduli space of semi-abelian varieties with torus part of rank $\leqslant 1$. It is independent of the choice of the compactification $\tilde{\mathcal{A}}_g$.

Let $j : \mathcal{A}_g \to \tilde{\mathcal{A}}_g$ be the inclusion map. We can rewrite the Eisenstein cohomology for a local system $\mathbb{V}_\lambda$ (both in Hodge cohomology and in étale cohomology) as

$$e_{\mathrm{Eis}}(\mathcal{A}_g,\mathbb{V}_\lambda) = e(\tilde{\mathcal{A}}_g, Rj_*\mathbb{V}_\lambda) - e(\tilde{\mathcal{A}}_g, Rj_!\mathbb{V}_\lambda),$$

where $j_!\mathbb{V}_\lambda$ is the extension by zero. This is a sum over strata

$$\sum_{j=0}^{g-1} e_c(q^{-1}(\mathcal{A}_j), Rj_*\mathbb{V}_\lambda - Rj_!\mathbb{V}_\lambda).$$

We define now the rank 1 part of the Eisenstein cohomology as the part that comes from the boundary component lying over $\mathcal{A}_{g-1}$:

$$e_{\mathrm{Eis},1}(\mathcal{A}_g,\mathbb{V}_\lambda) := e_c(q^{-1}(\mathcal{A}_{g-1}), Rj_*\mathbb{V}_\lambda - Rj_!\mathbb{V}_\lambda).$$

This contribution to the Eisenstein cohomology is independent of the compactification as $q^{-1}(\mathcal{A}_g \cup \mathcal{A}_{g-1})$ is so too.

Let $\mathbb{L} := h^2(\mathbb{P}^1)$ be the Lefschetz motive $h^2(\mathbb{P}^1)$ of rank 1 and weight 2. We call λ even if $\sum\lambda_i$ is even. Our result is an explicit formula for the rank 1 part of the Eisenstein cohomology.

Theorem 2.1. *For even λ the contribution $e_{\mathrm{Eis},1}(\mathcal{A}_g, \mathbb{V}_\lambda)$ to the Eisenstein cohomology of $\mathbb{V}_\lambda$ from the codimension 1 boundary is (both in Hodge cohomology and in étale cohomology) of the form*

$$\sum_{k=1}^{g} (-1)^k \, e_c(\mathcal{A}_{g-1}, \mathbb{V}_{\lambda_1+1,\lambda_2+1,\ldots,\lambda_{k-1}+1,\lambda_{k+1},\ldots,\lambda_g}) \, (1 - \mathbb{L}^{\lambda_k+g+1-k}).$$

Remark 2.2. The cohomology groups $H^i(\mathcal{A}_g, \mathbb{V}_\lambda)$ can be characterized inside the cohomology groups $H^{i+r}(\mathcal{Y}^s, \mathbb{C})$ with $\mathcal{Y}^r$ a Faltings-Chai compactification of the r-fold product of the universal abelian variety over $\mathcal{A}_g$ by using idempotents coming from the action of algebraic correspondences, see [9, p. 235]. This implies the fact that the result is formally the same for both Hodge and étale cohomology; cf. the discussion in [9, pp. 238–242].

Example 2.3. For $g = 1$ the cohomology of $\mathbb{V}_k$ can be only non-trivial for even k. In this case on gets for the Eisenstein cohomology $e_{\mathrm{Eis}}(\mathcal{A}_1, \mathbb{V}_k) = e_{\mathrm{Eis},1}(\mathcal{A}_1, \mathbb{V}_k)$ of $\mathbb{V}_k$ the polynomial $1 - \mathbb{L}^{k+1}$, in agreement with the Eichler-Shimura isomorphisms of [5] for $k > 0$ and Hodge cohomology

$$e_c(\mathcal{A}_1, \mathbb{V}_k \otimes \mathbb{C}) = -S_{k+2} \oplus \bar{S}_{k+2} - \mathbb{C}$$

or in motivic terms $e_c(\mathcal{A}_1, \mathbb{V}_k) = -S[k+2] - 1$ with $S[k+2]$ the motive associated by Scholl to the space S_{k+2} of cusp forms of weight $k+2$ (see [16]) and

$$e(\mathcal{A}_1, \mathbb{V}_k \otimes \mathbb{C}) = -S_{k+2} \oplus \bar{S}_{k+2} - \mathbb{C}(k+1)$$

or again $e(\mathcal{A}_1, \mathbb{V}_k) = -S[k+2] - \mathbb{L}^{k+1}$. Here S_k denotes the space of cusp forms of weight k on $\mathrm{SL}(2, \mathbb{Z})$. Together with the so-called congruence relation this shows that one can calculate the trace of the Hecke operator $T(p)$ on S_k by calculating the trace of Frobenius on $H^1_c(\mathcal{A}_1, \mathbb{V}_k)$ and subtracting 1. This 1 is the contribution from the Eisenstein cohomology. For $g > 2$ The Eisenstein cohomology is no longer so simple and our study of the Eisenstein cohomology arose from the desire to distill the traces of the Hecke operators from the traces of Frobenius on compactly supported cohomology.

Example 2.4. For $g = 2$, in view of the automorphism -1 on abelian surfaces cohomology of $\mathbb{V}_{l,m}$ can be only non-trivial for $l \equiv m \pmod 2$. In this case one gets for $e_{\mathrm{Eis},1}(\mathbb{V}_{l,m})$ the expression

$$e_c(\mathcal{A}_1, \mathbb{V}_{l+1})(1 - \mathbb{L}^{m+1}) - e_c(\mathcal{A}_1, \mathbb{V}_m)(1 - \mathbb{L}^{l+2}).$$

For $g = 2$ we also prove a formula for the *total* Eisenstein cohomology

$$\begin{aligned} e_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_{l,m}) = {} & -s_{l-m+2}(1 - \mathbb{L}^{l+m+3}) + s_{l+m+4}(\mathbb{L}^{m+1} - \mathbb{L}^{l+2}) \\ & + \begin{cases} e_c(\mathcal{A}_1, \mathbb{V}_m)(1 - \mathbb{L}^{l+2}) - (\mathbb{L}^{l+2} - \mathbb{L}^{l+m+3}), & l \text{ even}, \\ -e_c(\mathcal{A}_1, \mathbb{V}_{l+1})(1 - \mathbb{L}^{m+1}) - (1 - \mathbb{L}^{m+1}), & l \text{ odd}, \end{cases} \end{aligned}$$

where s_m denotes for $m > 2$ the dimension of the space of cusp forms of weight m on $\mathrm{SL}(2, \mathbb{Z})$ and $s_2 = -1$. From this formula one can deduce for *regular* λ (i.e., $l > m > 0$) the formula for the Eisenstein cohomology that was announced in the joint work with Faber [7], see Corollary 10.2. In that paper the term Eisenstein cohomology refers only to the kernel of $H_c^* \to H^*$.

Example 2.5. For $g = 3$ we get for $e_{\mathrm{Eis},1}$ the expression

$$e_c(\mathcal{A}_2, \mathbb{V}_{l+1,m+1})(1 - \mathbb{L}^{n+1}) - e_c(\mathcal{A}_2, \mathbb{V}_{l+1,n})(1 - \mathbb{L}^{m+2}) + e_c(\mathcal{A}_2, \mathbb{V}_{m,n})(1 - \mathbb{L}^{l+3}).$$

This formula together with numerical results suggested the conjectured formula of [2].

Remark 2.6. Note that the result of Theorem 2.1 is compatible with Poincaré duality, which says that

$$H^i(\mathcal{A}_g, \mathbb{V}_\lambda)^\vee \cong H_c^{2d-i}(\mathcal{A}_g, \mathbb{V}_\lambda^\vee(\nu^d)),$$

where $d = g(g+1)/2$. Here $\mathbb{V}(\nu)$ means the twist of $\mathbb{V}$ by the multiplier, cf. Section 3.

One may view the result here as an explicit formula for the general results of [14] for the symplectic group. In the work of Harder the non-vanishing of certain maps between cohomology groups and its relation with the critical values of L-series plays an important role. Here we are more interested in closed formulas for the Eisenstein formulas as our interest in Eisenstein cohomology arose in the joint work with Faber [7] where we tried to obtain information on Siegel modular forms by counting curves over finite fields. There Eisenstein cohomology contributes terms that one wants to remove, cf. [1, 7].

3 The group

Fix a positive integer g and let $V_{\mathbb{Z}}$ be the standard symplectic lattice of rank $2g$ with generators e_i and f_i $(i = 1, \ldots, g)$ with $\langle e_i, f_j \rangle = \delta_{ij}$, the Kronecker delta, for $1 \leqslant i, j \leqslant g$. We let $G = \mathrm{GSp}(2g)$ be the corresponding Chevalley group of symplectic similitudes of $V_{\mathbb{Z}}$. We shall write V for $\mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$.

An element $\gamma \in G$ can be written as $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ with a, b, c, d integral $g \times g$-matrices. For such a γ we write $a\,d^t - b\,c^t = \nu(\gamma)\,\mathrm{Id}_g$. Here $\nu : G \to \mathbf{G}_m$ is called the multiplier representation. It satisfies $\det = \nu^g : G \to \mathbf{G}_m$. We let G act on the left on V by matrix multiplication. We denote by M the subgroup of G of elements that respect the two subspaces $\langle e_i : i = 1, \ldots, g \rangle$ and $\langle f_i : i = 1, \ldots, g \rangle$. We can interpret the elements of M as matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $a \cdot d^t = \nu\, 1_g$. So $M \cong \mathrm{GL}(g) \times \mathbf{G}_m$. Furthermore, we let Q be the (maximal) parabolic subgroup of G that stabilizes the sublattice spanned by the vectors e_i $(i = 1, \ldots, g)$. Then $Q = M \ltimes U$ with $U = \mathrm{Hom}(\mathrm{Sym}^2(X), \mathbb{Z})$, the group of $\mathbb{Z}$-valued bilinear forms, or in terms of matrices, the group of matrices $\left(\begin{smallmatrix} 1_g & b \\ 0 & 1_g \end{smallmatrix}\right)$ in G. The standard maximal torus T of G can be identified with $\mathbb{G}_m^{g+1}$ via

$$(t_1, \ldots, t_g, x) \mapsto \mathrm{diag}(t_1, \ldots, t_g, x/t_1, \ldots, x/t_g).$$

The character group may thus be identified with the lattice

$$\Big\{(a_1, \ldots, a_g, c) \in \mathbb{Z}^{g+1} : \sum a_i \equiv c \,(\mathrm{mod}\ 2)\Big\},$$

via $\chi(\mathrm{diag}(t_1, \ldots, t_g, x/t_1, \ldots, x/t_g)) = x^{(c - \sum a_i)/2} \prod t_i^{a_i}$. We will sometimes view t_i $(i = 1, \ldots, g)$ and x as characters on T.

Let t be the complex Lie algebra of T. We have root systems Φ_G and Φ_M in $t^\vee$ and we choose compatible systems of positive roots Φ_G^+ and Φ_M^+. So

$$\Phi_M := \{(t_i/t_j)^{\pm} : 1 \leqslant i < j \leqslant g\},$$

and we set

$$\Psi_M = \{(t_i t_j/x)^{\pm} : 1 \leqslant i \leqslant j \leqslant g\}$$

so that

$$\Phi_G = \Phi_M \cup \Psi_M \quad \text{and} \quad \Phi_G^+ = \Phi_M^+ \cup \Psi_M^+$$

with

$$\Phi_M^+ = \{(t_i/t_j) : 1 \leqslant i < j \leqslant g\} \quad \text{and} \quad \Psi_M^+ = \{(t_i t_j/x) : 1 \leqslant i \leqslant j \leqslant g\}.$$

As usual we define $\rho = (1/2) \sum_{x \in \Phi_G^+} x$.

4 The final elements of the Weyl group

The Weyl group W_G of G is isomorphic to the semi-direct product $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, where the symmetric group S_g on g letters acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permuting the g factors. We interpret elements of W_G as signed permutations. The Weyl group W_M of M is isomorphic to the symmetric group S_g. They operate on the complex Lie algebra t and its dual $t^\vee$.

We define the set of Kostant representatives by

$$W^M = \{w \in W_G : \Phi_M^+ \subset w(\Phi_G)^+\},$$

or equivalently as

$$\{w \in W_G : \langle w(\rho) - \rho, u\rangle \geqslant 0 \quad \text{for all } u \in \Phi_M^+\}.$$

Concretely, for a sign change $\epsilon : (x_1, \ldots, x_g) \mapsto (\epsilon_1 x_1, \ldots, \epsilon_g x_g)$ with $\epsilon_i \in \{\pm 1\}$ there exists exactly one $\sigma \in S_g$ such that $\sigma\epsilon(\rho) - \rho$ is of the form $(a_1, \ldots, a_g)$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_g$ and this signed permutation is the Kostant representative in W^M. Note that $\rho = (g, g-1, \ldots, 2, 1, 0)$. Recall that W_G carries a length function ℓ.

Another way to describe the Weyl group (and the one we shall use in the following) is as the group of permutations

$$W_g := \{\sigma \in S_{2g} : \sigma(i) + \sigma(2g+1-i) = 2g+1 \, (i = 1, \ldots, g)\}.$$

Then the Weyl group of M can be identified with the subgroup

$$S_g = \{\sigma \in W_g : \sigma\{1, 2, \ldots, g\} = \{1, 2, \ldots, g\}\}.$$

The length $\ell(w)$ of an element $w \in W_g \subset S_{2g}$ is then defined by

$$\#\{i < j \leqslant g : w(i) > w(j)\} + \#\{i \leqslant j \leqslant g : w(i) + w(j) > 2g+1\}.$$

Lemma 4.1. *An element $\sigma \in W_g \subset S_{2g}$ is a Kostant representative if and only if $\sigma(i) < \sigma(j)$ for all $1 \leqslant i < j \leqslant g$.*

For the proof we refer to [6, Lemma 1]. In that paper the Kostant representatives are called it final elements. We shall adopt that usage here too. The set of 2^g final elements in the Weyl group is denoted by $F_g \subset S_{2g}$.

Lemma 4.2. *Let k be a natural number with $1 \leqslant k \leqslant g$. The set $\{\sigma \in F_g : \sigma^{-1}(k) \leqslant k\}$ (resp. the set $\{\sigma \in F_g : \sigma^{-1}(2g+1-k) \leqslant g\}$) has cardinality 2^{g-1} and can be identified in a natural way with F_{g-1} compatibly with the length function ℓ.*

Proof. Identify an element σ with its image g-tuple $[\sigma(1), \ldots, \sigma(g)]$. Let $A = \{\sigma \in F_g : \sigma^{-1}(k) \leqslant k\}$. For each $\sigma \in A$ we delete the entry equal to k from $[\sigma(1), \ldots, \sigma(g)]$. We rename the entries by replacing an entry m by $m-1$ if $k < m < 2g+1-k$ and by $m-2$ if $m > 2g+1-k$. We thus find the 2^{g-1} elements of F_{g-1} as one easily checks. The other statements are proved in a similar way. □

Example 4.3. We list the final elements and their lengths for $g = 3$ in Table 1. (For the third column see next two sections.)

5 Representations and vector bundles

Let t be the complex Lie algebra of T. The irreducible representations of G are parametrized by the characters of T that correspond to G-dominant (i.e., the scalar product with Φ_G^+ is non-negative) integral weights $\lambda \in t^\vee$. If λ is given by $(a_1, \ldots, a_g, c)$ this means that we have $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_g$. The standard representation corresponds to $(1, 0, \ldots, 0, 1)$. The irreducible representation associated with λ is denoted by $V(\lambda)$.

Table 1 Final elements for $g = 3$

$\ell(w)$	$[w(1), w(2), w(3)]$	$w(\lambda+\rho)-\rho$
0	[123]	(l, m, n)
1	[124]	$(l, m, -n-2)$
2	[135]	$(l, n-1, -m-3)$
3	[236]	$(m-1, n-1, -l-4)$
3	[145]	$(l, -n-3, -m-3)$
4	[246]	$(m-1, -n-3, -l-4)$
5	[356]	$(n-2, -m-4, -l-4)$
6	[456]	$(-n-4, -m-4, -l-4)$

We have

$$V(\lambda)^\vee = V(\lambda)\otimes \nu^k \text{ with } k \text{ equal to } \lambda \text{ evaluated at } -1_{2g}\in t.$$

The irreducible representations of M are parametrized by characters μ of T which are M-dominant (i.e., with non-negative scalar product with Φ_M^+) so that the representation $W(\mu)$ corresponding to μ has the highest weight μ.

Let $\mathcal{A}_g$ be the Deligne-Mumford stack of principally polarized abelian varieties of dimension g and let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ be the universal family. In the complex category we may identify $\mathcal{A}_g(\mathbb{C})$ with $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{D}$ with $\mathbb{D}$ the space of Lagrangian subspaces $W \subset V_{\mathbb{C}}$ on which $-\langle v,\bar{v}\rangle > 0$. The complex manifold $\mathbb{D}$ is contained in its so-called compact dual $\mathbb{D}^\vee = G/Q(\mathbb{C})$ of Lagrangian subspaces $W \subset V_{\mathbb{C}}$.

To each finite-dimensional complex representation r of $Q(\mathbb{C})$ on a vector space U one can associate a G-equivariant bundle E'_r on $\mathbb{D}^\vee$ defined as the quotient of $G(\mathbb{C})\times U$ under the equivalence relation

$$(gq, r(q)^{-1}u) \sim (g,u) \qquad \text{for all } g\in G(\mathbb{C}),\ q\in Q(\mathbb{C}).$$

Its restriction to $\mathbb{D}$ descends to a vector bundle E_r on $\mathcal{A}_g(\mathbb{C})$. If r is the restriction to $Q(\mathbb{C})$ of a finite-dimensional complex representation ρ of $G(\mathbb{C})$ then E_r carries a integrable connection defined as follows. An element $u \in U$, the fibre over the base point, defines a trivialization of E_r by $\gamma \mapsto \rho(\gamma)u$; hence an integrable connection on E_r and it descends to $\mathcal{A}_g(\mathbb{C})$.

The vector bundle associated with the standard representation of $G(\mathbb{C})$ can be identified with the relative de Rham homology of $\mathcal{X}_g$ and $R^1\pi_*(\mathbb{C})$ is the vector bundle associated with $\nu^{-1}\otimes$ the standard representation. The integrable connection is the Gauss-Manin connection.

After the choice of a base point we can view G as the fundamental group (arithmetic fundamental group) of $\mathcal{A}_g$. Therefore we can associate a local system (or a smooth $\mathbb{Q}_l$-sheaf) to each finite-dimensional representation of G. To the standard representation it associates the local system $R^1\pi_*\mathbb{Q}(1)$ (resp. $\mathbb{Q}_l$-sheaf $R^1\pi_*\mathbb{Q}_l(1)$ on $\mathcal{X}_g\otimes\mathbb{Z}[1/l]$). The character ν corresponds to $\mathbb{Q}(1)$ (or $\mathbb{Q}_\ell(1)$). We have a non-degenerate alternating pairing

$$R^1\pi_*\mathbb{Q}\times R^1\pi_*\mathbb{Q}\to\mathbb{Q}(-1).$$

The local system associated with such a λ is denoted by $\mathbb{V}_\lambda$. If λ is given by $(a_1,\ldots,a_g,c)$ this means that we have $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_g$. The local system corresponding to the standard representation defined by $(a_1,\ldots,a_g,c) = (1,0,\ldots,0,1)$ is $R^1\pi_*\mathbb{Q}(1)$. If we do not specify c then we assume that $c = \sum_{i=1}^g a_i$. Duality now says that we have a non-degenerate pairing

$$\mathbb{V}_\lambda\times\mathbb{V}_\lambda\to\mathbb{Q}(-|\lambda|)$$

with $|\lambda| = \sum\lambda_i$.

The irreducible representations of M are parametrized by characters μ of T which are M-dominant (i.e., with non-negative scalar product with Φ_M^+). To such a character we can associate a locally free $O_{\mathcal{A}_g}$-module (or vector bundle) W_μ. For example, the Hodge bundle of the universal family corresponds to the representation $\gamma = (a,b;0,d)\mapsto \nu(\gamma)^{-1}a$ acting on $\mathbb{C}^g$. Duality for W_μ says that $W_\mu^\vee = W_{-\sigma_1(w)}$ with σ_1 the longest element of S_g. Faltings showed that one can extend the vector bundles thus obtained to appropriate toroidal compactifications, cf. [9, Theorem 4.2].

6 The BGG complex

Let $\tilde{\mathcal{A}}_g$ be a Faltings-Chai compactification of $\mathcal{A}_g$ and let $j:\mathcal{A}_g\to\tilde{\mathcal{A}}_g$ be the natural inclusion and let $i: D\to\tilde{\mathcal{A}}_g$ be the inclusion of the divisor at infinity. Recall that D is a stack quotient of a Kuga-Satake variety, namely a compactified quotient of the universal abelian variety of dimension $g-1$ by the group $\mathrm{GL}(1,\mathbb{Z})$ which acts by $\{\pm1\}\in\mathrm{End}(X_\eta)$ on the generic fibre X_η.

According to Deligne the logarithmic de Rham complex w.r.t. the divisor D represents $Rj_*\mathbb{C}$. This generalizes for our sheaves $\mathbb{V}_\lambda$, where the role of the de Rham complex is played by the dual BGG complex which is obtained by applying ideas of Bernstein-Gelfand-Gelfand [3] to our situation as worked out in [9].

The dual BGG complex for λ is a direct summand of the de Rham complex for $\mathbb{V}_\lambda^\vee$ and consists of a complex $K_\lambda^\bullet$ of vector bundles on $\mathcal{A}_g$:

$$K_\lambda^q = \bigoplus_{w\in F_g,\ell(w)=q} W_{w*\lambda}^\vee,$$

where $w*\lambda = w(\lambda+\rho)-\rho$ and $W_\mu^\vee = W_{-\sigma_1(\mu)}$. This complex is a filtered resolution of $\mathbb{V}_\lambda^\vee$ on $\mathcal{A}_g$. The differentials of this complex are given by homogeneous differential operators on $\mathbb{D}^\vee$.

The vector bundles W_μ extend over the compactification $\tilde{\mathcal{A}}_g$ and so do the differential operators, resulting in a complex $\bar{K}_\lambda^\bullet$ on $\tilde{\mathcal{A}}_g$. We shall denote the extensions of W_μ again by the same symbol W_μ (or by $\bar{W}_\mu$ if confusion might arise). There is a variant of this complex $\bar{K}_\lambda^\bullet \otimes O_{\tilde{\mathcal{A}}_g}(-D)$ and the differentials extend also for this complex.

The filtration on the dual BGG complex induce decreasing filtrations on $\bar{K}_\lambda^\bullet$ and $\bar{K}_\lambda^\bullet \otimes O(-D)$ given by

$$F^p(\bar{K}_\lambda^\bullet) = \bigoplus_{w\in F_g, f(w,\lambda)\geqslant p} \bar{W}_{w*\lambda}^\vee,$$

where $f(w,\lambda) = (\sum\lambda_i + \sum\mu_i)/2$ with $\mu = -\sigma_1(w*\lambda)$, and in an analogous way for $\bar{K}_\lambda^\bullet \otimes O(-D)$. A term W_μ belongs to F^p if and only if $(\sum\lambda_i+\sum\mu_i)/2 \geqslant p$.

In [9, p. 233], it is shown that the (filtered) dual BGG complex $\bar{K}_\lambda^\bullet$ is quasi-isomorphic to $Rj_*\mathbb{V}_\lambda^\vee$, while $\bar{K}_\lambda^\bullet \otimes O(-D)$ is quasi-isomorphic to $Rj_!\mathbb{V}_\lambda^\vee$ and that the inclusion $\bar{K}_\lambda^\bullet \otimes O(-D) \subset \bar{K}_\lambda^\bullet$ corresponds to the natural map $Rj_!\mathbb{V}_\lambda^\vee \to Rj_*\mathbb{V}_\lambda^\vee$.

We have an exact sequence of complexes

$$0 \to \bar{K}_\lambda^\bullet \otimes O(-D) \to \bar{K}_\lambda^\bullet \to \bar{K}_\lambda^\bullet|D \to 0.$$

Therefore we can calculate the Eisenstein cohomology $e_{\mathrm{Eis}}(\mathcal{A}_g, \mathbb{V}_\lambda^\vee)$ by using the complex $\bar{K}_\lambda^\bullet|D = i^*\bar{K}_\lambda^\bullet$ obtained by restricting the dual BGG complex to D.

7 A calculation at the boundary

The boundary stratum D in $\tilde{\mathcal{A}}_g$ is a stratified space itself via the map q to the Satake compactification. The part $D' = D\cap\mathcal{A}_g'$ of D lying in $\mathcal{A}_g'$ has an étale cover $\mathcal{X}_{g-1}\to D'$ with $\mathcal{X}_{g-1}\to\mathcal{A}_{g-1}$ the universal principally polarized abelian variety of relative dimension $g-1$ and $\mathcal{X}_{g-1}\to\mathcal{A}_{g-1}$ factors through $q: D'\to\mathcal{A}_{g-1}/\mathrm{GL}(1,\mathbb{Z})$. The vector bundles W_μ extend canonically from D' to D.

We shall calculate the rank 1 part of the Eisenstein cohomology by using the Leray spectral sequence for the complex

$$\bigoplus_{w\in F_g} W_{w*\lambda}^\vee$$

by first restricting the factors $W_{w*\lambda}^\vee$ of the complex $\bigoplus_{w\in F_g} W_{w*\lambda}^\vee$ to D', extending these to a suitable Faltings-Chai compactification $\bar{D}'$ and then tensoring this complex with $O(-F)$ with F the divisor at infinity $\bar{D}'-D'$ of D' and by calculating the cohomology using the Leray spectral sequence for the map $q: D'\to\mathcal{A}_{g-1}/\mathrm{GL}(1,\mathbb{Z})$. (The tensoring with $O(-F)$ is done to get the rank-1 part of the Eisenstein cohomology and because of this the choice of the compactification of D' does not matter.)

As we shall now work at the same time with $\mathcal{A}_g$ and $\mathcal{A}_{g-1}$ we will write $W_\mu^{(g)}$ for W_μ on $\tilde{\mathcal{A}}_g$ in order to avoid confusion. We first do a computation on $\mathcal{X}_{g-1}\to\mathcal{A}_{g-1}$ and later take into account the action of $\mathrm{GL}(1,\mathbb{Z})$.

Proposition 7.1. *For $W_a = W_a^{(g)}$ with $a=(a_1,\ldots,a_g)$ on $\tilde{\mathcal{A}}_g$ we have*

$$\sum_j (-1)^j R^j q_*(W_a^{(g)}|\mathcal{X}_{g-1}) = \sum_{k=1}^{g} (-1)^{g-k} W^{(g-1)}_{(a_1,\ldots,a_{k-1},a_{k+1}-1,\ldots,a_g-1)}.$$

Proof. We consider the Hodge bundle $\mathbb{E}_g$ on $\tilde{\mathcal{A}}_g$. Its pullback to $\mathcal{X}_{g-1}$ fits into the exact sequence

$$0 \to q^*\mathbb{E}_{g-1} \to \mathbb{E}_g \to O_{\mathcal{X}_{g-1}} \to 0,$$

and we thus get

$$Rq_*\mathbb{E}_g = \mathbb{E}_{g-1} \otimes Rq_*O_{\mathcal{X}_{g-1}} + Rq_*O_{\mathcal{X}_{g-1}} = (\mathbb{E}_{g-1} + 1) \otimes \sum_{j=0}^{g-1} \wedge^j \mathbb{E}_{g-1}^\vee,$$

since $R^i q_* O_{\mathcal{X}_{g-1}} = \wedge^i R^1 q_* O_{\mathcal{X}_{g-1}}$. Note that

$$\sum_{j=0}^{g-1} \wedge^j \mathbb{E}_{g-1}^\vee = (-1)^k \sum_{k=0}^{g-1} W_{t_k}^{(g-1)}$$

with t_k denoting a vector $(0, \dots, 0, -1, \dots, -1)$ of length $g-1$ with $g-1-k$ zeros. Since $W_a^{(g)}$ is made by applying a Schur functor to the Hodge bundle the exact sequence for $\mathbb{E}_g$ implies

$$Rq_*W_a = \mathrm{Res}_{g-1}^g W_a^{(g)} \otimes \sum_{k=0}^{g-1} (-1)^k W_{t_k}^{(g-1)}.$$

Here $\mathrm{Res}_{g-1}^g W_a$ is the bundle obtained from restriction (branching) from $\mathrm{GL}(g)$ to $\mathrm{GL}(g-1)$. A well-known formula from representation theory (cf., e.g., [11]) says that we thus get

$$\mathrm{Res}_{g-1}^g W_a^{(g)} = \sum W_b^{(g-1)},$$

where the sum is over all (interlacing) $b = (b_1, \dots, b_{g-1})$ with $a_1 \geqslant b_1 \geqslant a_2 \geqslant \cdots \geqslant b_{g-1} \geqslant a_g$. Moreover, we have that $W^{(g-1)}(b) \otimes \sum_{k=0}^{g-1} (-1)^k W_{t_k}^{(g-1)}$ is a signed sum of $W_{b'}^{(g-1)}$'s with the sum running over all vectors $b' \in \mathbb{Z}^{g-1}$ obtained from subtracting a 1 from k entries b_i with k between 0 and $g-1$ and deleting those b' that does not satisfy the condition that $b'_i \geqslant b'_{i+1}$. Carrying out the summation we see that most terms telescope away and what remains is the right-hand side of the statement in the proposition. □

For example, for $g = 2$ we get that $Rq_*W_{a,b,c}^{(3)}$ is equal to

$$\sum_{\alpha,\beta} W_{\alpha,\beta}^{(2)} - W_{\alpha,\beta-1}^{(2)} - W_{\alpha-1,\beta}^{(2)} + W_{\alpha-1,\beta-1}^{(2)},$$

where the sum is over all (α, β) with $a \geqslant \alpha \geqslant b \geqslant \beta \geqslant c$ and $W_2(r, s) = 0$ if $r < s$. What remains is $W_2(a, b) - W_2(a, c-1) + W_2(b-1, c-1)$.

However, we still need to take the action of $\mathrm{GL}(1, \mathbb{Z})$ into account. The non-trivial elements act by -1 on the fibres of q. Therefore only the terms in the right-hand side of Proposition 7.1 which are even (in the sense that $\sum_{i=1}^{k-1} a_i + \sum_{i=k+1}^{g} (a_i - 1) \equiv 0 \pmod 2$) will contribute to

$$\sum_j (-1)^j R^j q_*(W_a^{(g)} | D').$$

Recall that we write $w * \lambda$ for the operation $\lambda \mapsto w(\lambda + \rho) - \rho$ of W_g on the set of lambda's. Recall also we have $W(w * \lambda)^\vee = W(-\sigma_1(w * \lambda))$ with σ_1 the longest element of S_g.

Our Eisenstein cohomology is now given by a complex which is a sum over $w \in F_g$ of terms $Rq_*W_{w*\lambda}^\vee$. By Proposition 7.1 the term $Rq_*W_{w*\lambda}^\vee$ yields a complex

$$\sum_{l=1}^{g} (-1)^l W_{\tau_l(-\sigma(w*\lambda))}^{(g-1)},$$

where τ_l applied to a vector $(a_1, \dots, a_g)$ is the vector $(a_1, \dots, a_{l-1}, a_{l+1} - 1, \dots, a_g - 1)$. We thus get a double sum $\sum_{w \in F_g} \sum_{l=1}^{g} X_{w,l}$ of terms $X_{w,l}$ (that are (signed) sheaves of the form W_μ) which we rewrite as a sum of two double sums

$$\sum_{k=1}^{g} \sum_{w \in F_g, w^{-1}(k) \leqslant k} X_{w,k} + \sum_{k=1}^{g} \sum_{w \in F_g, w^{-1}(2g+1-k) \leqslant g} X_{w,k}.$$

By Lemma 4.2 the inner sum $\sum_{w\in F_g, w^{-1}(k)\leqslant k} X_{w,k}$ in the first double sum is equal to

$$\sum_{u\in F_{g-1}} (-1)^k (W^{(g-1)}_{u*\tau'_k(\lambda)})^\vee,$$

where $\tau'_k(a_1,\ldots,a_g)=(a_1+1,\ldots,a_{k-1}+1,a_{k+1},\ldots,a_g)$. The double sum $\sum_{k=1}^g \sum_{w\in F_g,w^{-1}(k)\leqslant k} X_{w,k}$ thus contributes the complex

$$K'_\lambda := \sum_{k=1}^g (-1)^k K_{\tau'_k(\lambda)}.$$

The Hodge weight of these terms can be read off from Lemma 4.2 and Faltings' results and thus contribute the cohomology of $\sum_k (-1)^k \mathbb{V}_{\tau'_k(\lambda)}$. (See the diagram in Section 10 for an illustration in case $g=2$.) In view of duality (Poincaré and Serre duality, see [9, p. 236]) the remaining $2^{g-1}g$ terms of the sum $\sum_{k=1}^g \sum_{w\in F_g,w^{-1}(2g+1-k)\leqslant g} X_{w,k}$ will contribute the dual terms

$$\sum_{k=1}^g (-1)^{k+1} K_{\tau'_k(\lambda)} \otimes \nu^{\lambda_k+g+1-k}$$

and this contributes the cohomology of $\sum_k (-1)^{k+1}\mathbb{V}_{\tau'_k(\lambda)}$ twisted by the power $\mathbb{L}^{g+1+\lambda_k-k}$ of $\mathbb{L}$.

But we need the rank 1 part. To get this we consider the divisor Δ at infinity of $\tilde{\mathcal{A}}_{g-1}$ (where $\tilde{\mathcal{A}}_{g-1}$ is defined as the closure of the zero-section of $\mathcal{X}_{g-1}\to\mathcal{A}_{g-1}$) and take $\bar{K}'_\lambda\otimes O(-\Delta)$ instead of K'_λ. Here $\bar{K}'_\lambda$ is the extension over $\tilde{\mathcal{A}}_{g-1}$ which we know to exist. Then our Eisenstein cohomology is of the form

$$\sum_{k=1}^g (-1)^k e_c(\mathcal{A}_{g-1}, \mathbb{V}_{\tau'_k(\lambda)})(1-\mathbb{L}^{g+1+\lambda_k-k}).$$

In the next section we show that this is compatible with the action of the Hecke algebras.

8 The action of the Hecke operators

The Hecke algebra $\mathcal{H}_0(\mathrm{GSp}(2g,\mathbb{Q}),\mathbb{Q})$ acts on the cohomology $H^*(\mathcal{A}_g,\mathbb{V}_\lambda)$ and $H^*_c(\mathcal{A}_g,\mathbb{V}_\lambda)$ as explained in [9, Ch. VII]. It also acts in a compatible way on the dual BGG complex. These operators are defined by algebraic cycles and this guarantees that they respect the mixed Hodge structure on the Betti cohomology and the Galois structure on étale cohomology. Moreover, they are self-adjoint for Serre and Poincaré duality.

The compatible action of the Hecke operators on both $H^*(\mathcal{A}_g,\mathbb{V}_\lambda)$ and $H^*_c(\mathcal{A}_g,\mathbb{V}_\lambda)$ induces an action on the (total) Eisenstein cohomology. We can see this action by means of its action on the dual BGG complex and its restriction to the boundary D. We now show that it factors through the action of the Hecke algebra for $\mathrm{GSp}(2g-2,\mathbb{Z})$.

The correspondences $T\to\mathcal{A}_g\times\mathcal{A}_g$ that define the Hecke operators extend in a natural way to $\mathcal{A}'_g\times\mathcal{A}'_g$. The (pullback of the) restriction of such a T to $\mathcal{X}_{g-1}\times\mathcal{X}_{g-1}$, a cover of $D'\times D'$, is given by $T'\to\mathcal{X}_{g-1}\times\mathcal{X}_{g-1}$ which lies over a component of a Hecke correspondence $T''\to\mathcal{A}_{g-1}\times\mathcal{A}_{g-1}$. In the next paragraph we indicate this for the complex case.

Proposition 8.1. *The action of the Hecke algebra $\mathcal{H}_0(\mathrm{GSp}(2g,\mathbb{Q}),\mathbb{Q})$ on the rank 1 Eisenstein cohomology induces an action of $\mathcal{H}_0(\mathrm{GSp}(2g-2,\mathbb{Q}),\mathbb{Q})$.*

Proof. The map $T'\to T''$ is a universal family of abelian varieties. The action is on bundles that are pullbacks from $\mathcal{A}_{g-1}$. The action of T'/T'' is defined by a correspondence on the abelian variety that is the fibre over $\mathcal{A}_{g-1}$. The generic abelian variety over $\mathcal{A}_{g-1}$ has only $\mathbb{Z}$ as non-trivial endomorphisms. We thus see that the action of T'/T'' is induced by an element of $\mathbb{Z}\subset\mathrm{End}(X_\eta)$ with X_η, an abelian variety, the generic fibre of T' over T'', on the cohomology of the stucture sheaf O_{X_η}. This action is by scalars. The action of T'' belongs to the action of the Hecke algebra of $\mathrm{GSp}(2g-2,\mathbb{Z})$. Hence the Hecke algebra of $\mathrm{GSp}(2g,\mathbb{Z})$ on the Eisenstein cohomology factors through an action of the Hecke algebra of $\mathrm{GSp}(2g-2,\mathbb{Z})$. □

We illustrate this over $\mathbb{C}$. A component of a Hecke correspondence is defined by an embedding $\mathcal{H}_g \to \mathcal{H}_g \times \mathcal{H}_g$, with $\mathcal{H}_g$ the Siegel upper half plane, and given by equations

$$wcz + wd - az - b = 0, \tag{1}$$

where w, z are in $\mathcal{H}_g$ and $(a, b; c, d)$ is an integral $2g\times 2g$-matrix which lies in $\mathrm{GSp}(2g,\mathbb{Q})$. This component can be extended to a correspondence for $\mathcal{A}'_g$ that restricts to $T' \to \mathcal{X}_{g-1} \times \mathcal{X}_{g-1}$ given by an embedding of $\mathcal{H}_{g-1} \times \mathbb{C}^{g-1}$ into $(\mathcal{H}_{g-1} \times \mathbb{C}^{g-1})^2$. We consider the behavior at infinity given by

$$\lim_{t\to\infty} \begin{pmatrix} z' & \zeta \\ \zeta^t & it \end{pmatrix}, \qquad z' \in \mathcal{H}_{g-1}, \zeta \in \mathbb{C}^{g-1}.$$

In order that the component given by (1) does intersect our boundary component the matrix $(a, b; c, d)$ must have the form

$$\begin{pmatrix} a' & 0 & b' & * \\ * & u & * & * \\ c' & 0 & d' & * \\ 0 & 0 & 0 & u^{-1} \end{pmatrix}, \qquad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{Sp}(2g-2,\mathbb{Q}), \quad u \in \mathbb{Q}^*.$$

This correspondence in $\mathcal{X}_{g-1} \times \mathcal{X}_{g-1}$ lies over a component of a Hecke correspondence $T'' \to \mathcal{A}_{g-1} \times \mathcal{A}_{g-1}$ given by

$$w'c'z' + w'd' - a'z' - b' = 0, \qquad z', w' \in \mathcal{H}_{g-1}.$$

and in the fibres it is given by

$$\eta^t(c'z' + d') = u\zeta + (\alpha z' + \beta),$$

where η is the analogue for w of ζ for z and α and β are appropriate integral matrices.

9 An example: $g = 1$

We consider the case of a local system $\mathbb{V}_k = \mathrm{Sym}^k(\mathbb{V})$ for k even. For $g = 1$ the BGG complex is $0 \to j_*\mathbb{V}_k^\vee \to W_{-k} \to W_{k+2} \to 0$ with $\mathbb{V}_k = \mathrm{Sym}^k(R^1\pi_*(\mathbb{Q})(1))$. Similarly, there is a complex $0 \to j_!\mathbb{V}_k^\vee \to W_{-k}(-D) \to W_{k+2}(-D) \to 0$ with D the divisor $\tilde{\mathcal{A}}_1 - \mathcal{A}_1$ at infinity. By the exact sequence

$$0 \to W_m(-D) \to W_m \to W_m|D \to 0,$$

we get

$$e(j_*\mathbb{V}_k^\vee) - e(j_!\mathbb{V}_k^\vee) = e(D, W_{-k}|D) - e(D, W_{k+2}|D).$$

The bundle W_a is associated with a representation of Q, the parabolic subgroup. The action of the central multiplicative group is not trivial: W_1 is associated with the representation where

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ acts by multiplication by } 1/d \text{ on } \mathbb{C}.$$

In view of $\mathbb{V}_k^\vee = \mathbb{V}_k \otimes \nu^k$ we see that $e_{\mathrm{Eis}}(\mathbb{V}_k)$ is equal to $e(D, W'_{-k}|D) - e(D, W'_{k+2}|D)$, where the prime refers to k times twisting. This implies that $W'_c|D$ is $\mathbb{C}((k + c)/2)$ on D and that

$$e_{\mathrm{Eis}}(\mathbb{V}_k) = \mathbb{L}^0 - \mathbb{L}^{k+1}$$

for $k \geqslant 0$ even.

Our answer for the Eisenstein cohomology is compatible with Poincaré duality. It has a part with Hodge weight $k+1$ and one with Hodge weight 0. The part of Hodge weight $k+1$ occurs for $k \geqslant 2$ in the exact sequence

$$0 \to H^0(W_{k+2} \otimes O(-D)) \to H^0(W_{k+2}) \to H^0(D, W_{k+2}|D) \to 0.$$

This can be identified with

$$0 \to S_{k+2} \to M_{k+2} \to \mathbb{C}(k+1) \to 0,$$

with M_{k+2} (resp., S_{k+2}) the space of modular forms (resp., of cusp forms) of weight $k+2$ on $\mathrm{SL}(2,\mathbb{Z})$. The other part occurs in

$$0 \to H^0(D, W_{-k}|D) \to H^1(W_{-k} \otimes O(-D)) \to H^1(W_{-k}) \to 0.$$

All in all we get

$$e_c(\mathcal{A}_1, \mathbb{V}_k) = -S_{k+2} \oplus \bar{S}_{k+2} - 1$$

and

$$e(\mathcal{A}_1, \mathbb{V}_k) = -S_{k+2} \oplus \bar{S}_{k+2} - \mathbb{C}(k+1).$$

We identify $\mathbb{C}$ and $\mathbb{C}(k+1)$ with the Eisenstein cohomology. Note that if we let $W'_m = W_m \otimes O(-\Delta)$ with Δ the divisor on D that is the fibre over the cusp ∞ of $\tilde{\mathcal{A}}_1$ we also have the identities

$$\begin{aligned} e_c(\mathcal{A}_1, \mathbb{V}_k) &= -e(\tilde{\mathcal{A}}_1, W'_{k+2}) + e(\tilde{\mathcal{A}}_1, W'_{-k}), \\ e(\mathcal{A}_1, \mathbb{V}_k) &= -e(\tilde{\mathcal{A}}_1, W_{k+2}) + e(\tilde{\mathcal{A}}_1, W_{-k}). \end{aligned}$$

For even $k \geqslant 4$ we let $S[k]$ denote the motive of cusp forms of weight k on $\mathrm{SL}(2,\mathbb{Z})$ as constructed by Scholl, cf. [16], see also [4]. For $k=2$ we put $S[2] = -\mathbb{L} - 1$. In the category of Hodge structures we have for $k \geqslant 2$

$$S[k+2] = e(\tilde{\mathcal{A}}_1, W'_{k+2}) - e(\tilde{\mathcal{A}}_1, W_{-k}).$$

Note that by Serre duality we have $H^1(\tilde{\mathcal{A}}_1, W_{-k}) \cong H^0(\tilde{\mathcal{A}}_1, \Omega^1 \otimes W_k)^\vee = H^0(\tilde{\mathcal{A}}_1, W'_{k+2})^\vee$ but if we take into account the action of the central $\mathbb{G}_m$ we have to twist by η^{k+1}. We now have for even $k \geqslant 0$

$$e_c(\mathcal{A}_1, \mathbb{V}_k) = -S[k+2] - 1, \qquad e(\mathcal{A}_1, \mathbb{V}_k) = -S[k+2] - \mathbb{L}^{k+1}.$$

10 The genus 2 case

We now look at the case $g=2$ and consider the local system $\mathbb{V}_{l,m}$ with $l \equiv m \pmod 2$. Calculations with Faber [7] led to the formulas for this case. We have a standard Faltings-Chai compactification $\tilde{\mathcal{A}}_2$ in this case which coincides with Igusa's blow-up of the Satake compactification and also with the moduli space $\overline{\mathcal{M}}_2$ of stable curves of genus 2. We consider the full (or total) Eisenstein cohomology

$$e_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_{l,m}) := e(\mathcal{A}_2, Rj_* \mathbb{V}_{l,m}) - e(\mathcal{A}_2, Rj_! \mathbb{V}_{l,m}).$$

Theorem 10.1. *The Eisenstein cohomology $e_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_{l,m})$ is given by*

$$-s_{l-m+2}(1-\mathbb{L}^{l+m+3}) + s_{l+m+4}(\mathbb{L}^{m+1} - \mathbb{L}^{l+2}) + \begin{cases} e_c(\mathcal{A}_1, (\mathbb{V}_m)(1-\mathbb{L}^{l+2}) - (\mathbb{L}^{l+2} - \mathbb{L}^{l+m+3}), & l \text{ even}, \\ -e_c(\mathcal{A}_1, \mathbb{V}_{l+1})(1-\mathbb{L}^{m+1}) - (1-\mathbb{L}^{m+1}), & l \text{ odd}. \end{cases}$$

Alternatively, the Eisenstein cohomology can be written as

$$-(s_{l-m+2}+1)(1-\mathbb{L}^{l+m+3}) + s_{l+m+4}(\mathbb{L}^{m+1} - \mathbb{L}^{l+2}) + \begin{cases} -S[m+2](1-\mathbb{L}^{l+2}) & l \text{ even}, \\ S[l+3](1-\mathbb{L}^{m+1}) & l \text{ odd}. \end{cases}$$

We can check this for example for $l = m = 0$. We have

$$e(Rj_*\mathbb{V}_{0,0}) - e(Rj_!\mathbb{V}_{0,0}) = 1 + \mathbb{L} - \mathbb{L}^2 - \mathbb{L}^3,$$

while for the *compactly supported Eisenstein cohomology* (i.e., the kernel of $H_c^* \to H^*$) we find $1 + \mathbb{L}$, which fits. If $\mathbb{V}_{l,m}$ is a regular local system (i.e., $l > m > 0$) then the Hodge weights of the terms are either $> l+m+3$ or $< l+m+3$ and this determines whether the term belongs to compactly supported Eisenstein cohomology, cf. [15, Theorem 3.5]. We thus get the result announced in [7].

Corollary 10.2. [7] *The compactly supported Eisenstein cohomology for a regular local system* $\mathbb{V}_{l,m}$ *is given by*

$$s_{l-m+2} - s_{l+m+4}\mathbb{L}^{m+1} + \begin{cases} S[m+2]+1, & l \text{ even}, \\ -S[l+3], & l \text{ odd}. \end{cases}$$

In [7] one finds numerical confirmation of these formulas. The BGG complex for $j_*\mathbb{V}_{l,m}^\vee$ is

$$0 \to j_*\mathbb{V}_{l,m}^\vee \to W_{-m,-l} \to W_{m+2,-l} \to W_{l+3,1-m} \to W_{l+3,m+3} \to 0$$

and for the compactly supported cohomology the similar complex is

$$0 \to j_!\mathbb{V}_{l,m}^\vee \to W_{-m,-l} \otimes O(-D) \to \dots$$

The extended complexes over $\tilde{\mathcal{A}}_g$ are quasi-isomorphic to $Rj_*(\mathbb{V}_\lambda)^\vee$ and $Rj_!(\mathbb{V}_\lambda)^\vee$, see [9, Proposition 5.4] and above. We have to twist $l+m$ times if we work with $Rj_*(\mathbb{V}_\lambda)$ and $Rj_!(\mathbb{V}_\lambda)$. We hope that the details of the first part of the proof in this case will illustrate the proof of the formula for $e_{\mathrm{Eis},1}$ for the general case. The short exact sequence

$$0 \to W_\mu(-D) \to W_\mu \to W_\mu|D \to 0$$

yields

$$\begin{aligned} e(Rj_*\mathbb{V}_{l,m}) - e(j_!\mathbb{V}_{l,m}) &= e(W_{-m,-l}|D) - e(W_{m+2,-l}|D) \\ &\quad + e(W_{l+3,1-m}|D) - e(W_{l+3,m+3}|D)) \end{aligned}$$

if we take into account an $(l+m)$-th twist. Therefore we first determine the Euler characteristic $e(W_{a,b}|D)$. Note that D is the quotient by -1 of the compactification $\tilde{\mathcal{X}}_1 \to \tilde{\mathcal{A}}_1$ of the universal elliptic curve $\mathcal{X}_1 \to \mathcal{A}_1$. We stratify $\tilde{\mathcal{X}}_1$ by the open part and the fibre F over the cusp ∞ of $\tilde{\mathcal{A}}_1$. The cohomology

$$e\left(\tilde{\mathcal{A}}_1, \sum_j (-1)^j R^j q_*(W_\mu|\mathcal{X}_1)\right)$$

can be calculated via the exact sequence for the Hodge bundle $\mathbb{E}_2 = W_{1,0}$,

$$0 \to q^*\mathbb{E}_1 \to \mathbb{E}_2|\mathcal{X}_1 \to O_{\mathcal{X}_1} \to 0.$$

More generally, the pull back of $W_{a,b}$ to $\mathcal{X}_1$ is $\mathrm{Sym}^{a-b}\mathbb{E}_1 \otimes \det \mathbb{E}_1^b$ with $\mathbb{E}_1$ the Hodge bundle. But we need to keep track of the twisting. The exact sequence above gives (using that $R^i q_* O_{\mathcal{X}_1} = \wedge^i \mathbb{E}_1^\vee$)

$$Rq_* W_{a,b}|\mathcal{X}_1 = \sum_{a \geqslant \nu \geqslant b} W_\nu \otimes (1 - \mathbb{E}_1^\vee)$$

and carries out the summation we find

$$Rq_* W_{a,b}|\mathcal{X}_1 = W_a - W_{b-1}.$$

We now replace the W_μ's in our calculation by $W'_\mu := W_\mu(-F)$ and then calculate separately the contribution from the stratum F. We collect the terms in Table 2. An element $w \in F_2 \subset S_4$ is identified with its images $[w(1)w(2)]$ of the elements 1 and 2.

Table 2

w	$\ell(w)$	$-\sigma_1(w*\lambda)$	Contribution
[12]	0	$(-m,-l)$	$W'_{-m}\ -W'_{-l-1}$
[13]	1	$(m+2,-l)$	$-W'_{m+2}\ +W'_{-l-1}(\nu^{m+1})$
[24]	2	$(l+3,1-m)$	$W'_{l+3}\ -W'_{-m}(\nu^{l+2})$
[34]	3	$(l+3,m+3)$	$-W'_{l+3}(\nu^{m+1})\ +W'_{m+2}(\nu^{l+2})$

We can collect this into $-e_c(\mathcal{A}_1,\mathbb{V}_m)(1-\mathbb{L}^{l+2})+e_c(\mathcal{A}_1,\mathbb{V}_{l+1})(1-\mathbb{L}^{m+1})$.

We now treat the codimension 2 boundary contribution. It is a priori clear that the outcome will be a polynomial in $\mathbb{L}$ as we are calculating over a toric curve; in fact one can also deduce that the exponents of $\mathbb{L}$ that occur in $\{l+m+3,l+2,m+1,0\}$.

Let F be the fibre of q over $\tilde{\mathcal{A}}_0$. Note that F has dimension 1. The neighborhood of F in the toroidal compactification $\tilde{\mathcal{A}}_2$ is a toric variety obtained by glueing infinitely many copies of affine 3-space A^3 and dividing through an action of $\mathrm{GL}(2,\mathbb{Z})$; more precisely it is an orbifold constructed as follows. The cone of symmetric real positive definite 2×2 matrices has a natural cone decomposition invariant under the action of $\mathrm{GL}(2,\mathbb{Z})$. The group $\mathrm{GL}(2,\mathbb{Z})$ acts on the cone of positive definite 2×2 matrices by

$$C=\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\mapsto A^tCA.$$

The cone decomposition consists of the orbit under $\mathrm{GL}(2,\mathbb{Z})$ of the cone spanned by the three matrices

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

Associated with this we have the toroidal variety $\mathcal{T}$ with $\mathrm{GL}(2,\mathbb{Z})$-action. The coordinate axes in the A^3's are glued to a union of $\mathbb{P}^1$'s. The 1-skeleton $\mathcal{T}^1$ of the quotient of this toric variety under $\mathrm{GL}(2,\mathbb{Z})$ can be identified with F. In general in level $n\geqslant 3$ we have $(1/4)n^3\prod_{p|n}(1-p^{-2})$ copies of $\mathbb{P}^1$ meeting three at each one of the $(1/6)n^3\prod_{p|n}(1-p^{-2})$ points. For $n=3$ this looks like a tetrahedron, a cube for $n=4$ etc. and in general as the polyhedral decomposition of the Riemann surface $\Gamma(n)\backslash\mathcal{H}_1$ with $\Gamma(n)$ the full level n congruence subgroup.

The associated quadratic form $\alpha x^2+2\beta xy+\gamma y^2$ determines a point $z=(-\beta+\sqrt{\beta^2-\alpha\gamma})/\alpha$ in the upper half plane (this factors through scaling by positive reals). The action of $\mathrm{diag}(1,-1)$ is given by $\beta\mapsto-\beta$ and induces $z\mapsto-\bar{z}$ on the upper half plane.

In order to calculate the complex $\bar{K}_\lambda|F$ we restrict the Hodge bundle $\mathbb{E}$ to F. Note that the fundamental domain for the action of $\Gamma[n]\subset\mathrm{GL}(2,\mathbb{Z})$ is a cone over a fundamental domain for the modular curve of level n.

The restriction of the Hodge bundle to a $\mathbb{P}^1$ in F is of the form $O_{\mathbb{P}^1}\oplus O_{\mathbb{P}^1}$. The pullback of $\mathbb{E}$ to $\mathcal{T}^1$ is flat vector bundle of rank 2 determined by the standard representation of $\mathrm{GL}(2)$. So the bundle $W(a,b)$ is associated to the irreducible representation of type (a,b) of $\mathrm{GL}(2)$. This and the toroidal construction makes it possible to express the cohomology in terms of group cohomology.

Proposition 10.3. *Let*

$$U=W_{-m,-l}-W_{m+2,-l}+W_{l+3,1-m}-W_{l+3,m+3}.$$

Then the Euler characteristic $e_c(F,U|F)$ equals

$$-s_{l-m+2}(\mathbb{L}^0-\mathbb{L}^{l+m+3})+s_{l+m+4}(\mathbb{L}^{m+1}-\mathbb{L}^{l+2})+\begin{cases}-\mathbb{L}^{l+2}+\mathbb{L}^{l+m+3}, & l\ \text{even},\\ -1+\mathbb{L}^{m+1}, & l\ \text{odd}.\end{cases}$$

Proof. The Euler characteristic $e_c(F, W(a,b))$ can be expressed in group cohomology. We calculate it by using the stratification of F by the open part and the cusp. We then get the Euler characteristic of compactly supported cohomology of $\mathrm{GL}(2,\mathbb{Z})$ with values in V_{a-b}. This gives for the four cases $(-m,-l),\ldots,(l+3,m+3)$ the contributions $(-s_{l-m+2}-1)\mathbb{L}^x$, $(-s_{l+m+4}-1)\mathbb{L}^y$, $(s_{l+m+4}+1)\mathbb{L}^{l+m+3-y}$ and $(s_{l-m+2}+1)\mathbb{L}^{l+m+3-x}$ for appropriate x and y which turn out to be 0 and $m+1$. We have to add the contribution from the cusp. This gives $\mathbb{L}^{l+2}-\mathbb{L}^{l+m+3}$ for l odd and $1-\mathbb{L}^{m+1}$ for l even. For this just look at the action of $\mathrm{diag}(1,-1)$. If ℓ (and hence m) is even only the first two terms ($W_{-m,-l}$ and $-W_{m+2,-l}$) contribute because of the sign, while for the odd case the two other terms contribute. □

Remark 10.4. The same method suffices to prove the analogues for the moduli space $\mathcal{A}_2[n]$ of abelian surfaces with a level n structure.

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References

1 Bergström J, Faber C, van der Geer G. Siegel modular forms of genus 2 and level 2: cohomological computations and conjectures. Int Math Res Not IMRN, 2008, doi: 10.1093/imrn/rnn100
2 Bergström J, Faber C, van der Geer G. Siegel modular forms of genus 3. In preparation
3 Bernstein I N, Gelfand I M, Gelfand S I. Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. In: Lie groups and their representations. I.M. Gelfand ed. London: Adam Hilger, 1975
4 Consani K, Faber C. On the cusp form motives in genus 1 and level 1. In: Moduli Spaces and Arithmetic Geometry. Advanced Studies in Pure Mathematics, 2006
5 Deligne P. Formes modulaires et représentations ℓ-adiques. Séminaire Bourbaki, 1968-1969, exp. 355
6 Ekedahl T, van der Geer G. Cycle Classes of the E-O Stratification on the Moduli of Abelian Varieties. arXiv:math/0412272. In: Algebra, Arithmetic and Geometry-Manin Festschrift. Progress in Math 269. Boston: Birkhäuser Verlag, 2009, 567–636
7 Faber C, van der Geer G. Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes, I, II. C R Acad Sci Paris, Sér I, 2004, 338: 381–384, 467–470
8 Faltings G. On the cohomology of locally symmetric Hermitian spaces. Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year, Lecture Notes in Math, vol. 1029. Berlin: Springer, 1983, 55–98
9 Faltings G, Chai C L. Degeneration of Abelian Varieties. In: Ergeb Math Grenzgeb, 22. Berlin: Springer, 1990
10 Franke J. Harmonic analysis in weighted L_2-spaces. Ann Sci École Norm Sup, 1998, 31: 181–279
11 Fulton W, Harris J. Representation Theory. A First Course. New York: Springer-Verlag, 1991
12 Harder G. Eisenstein cohomology of arithmetic groups. The case GL_2. Invent Math, 1987, 89: 37–118
13 Harder G. Eisensteinkohomologie und die Konstruktion gemischter Motive. Springer Lecture Notes in Mathematics 1562. Berlin: Springer, 1993
14 Pink R. On ℓ-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. Math Ann, 1992, 292: 197–240
15 Peters C, Saito M. Lowest weights in Cohomology of Variations of Hodge structure. Preprint February 2008, arXiv: 0708.0130v2
16 Scholl A J. Motives for modular forms. Invent Math, 1990, 100: 419–430
17 Schwermer J. On Euler products and residual Eisensteincohomology classes of Siegel modular varieties. Forum Math, 1995, 7: 1–28
18 Schwermer J. On arithmetic quotients of the Siegel upper half space of degree two. Compos Math, 1986, 58: 233–258