CYCLES REPRESENTING THE TOP CHERN CLASS OF THE HODGE BUNDLE ON THE MODULI SPACE OF ABELIAN VARIETIES

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Abstract
We give a generalization to higher genera of the famous formula $12\,\lambda = \delta$ *for genus* 1.

1. Introduction

The fact that there exists a cusp form of weight 12 on $\mathrm{SL}(2,\mathbb{Z})$ with a simple zero at the cusp and no zero on the upper half-plane translates into the cycle relation $12\lambda = \delta$ (in the Chow ring of the moduli space of semistable curves of genus 1). Here λ is the divisor class corresponding to the factor of automorphy

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto (cz+d),$$

and δ represents the class of the cusp. We wish to generalize this relation to a relation in the Chow ring (but now with rational coefficients) of the moduli of principally polarized abelian varieties. The analogue of the class λ is the top Chern class λ_g of the Hodge bundle, and the analogue of δ is a codimension g class δ_g living on the boundary of the moduli space. The analogue of the formula is then a relation of the form

$$\lambda_g = (-1)^g \zeta(1-2g)\,\delta_g$$

with $\zeta(s)$ the Riemann zeta function. We now formulate a precise version of this.

Let $\mathscr{A}_g/\mathbb{Z}$ denote the moduli stack of principally polarized abelian varieties of dimension g. This is an irreducible algebraic stack of relative dimension $g(g+1)/2$ with irreducible fibres over $\mathbb{Z}$. The stack $\mathscr{A}_g$ carries a locally free sheaf $\mathbb{E}$ of rank g, the Hodge bundle, defined as follows. If $\pi : A \to S$ is an abelian scheme over S with zero section s, we get a locally free sheaf $s^*\Omega^1_{A/S}$ of rank g on S. This is compatible with pullbacks and gives $\mathbb{E}$ on $\mathscr{A}_g$. The bundle $\Omega^1_{A/S}$ is isomorphic to the pullback under π of the Hodge bundle. The top Chern class $\lambda_g(A/S) := c_g(s^*\Omega^1_{A/S})$ (in the

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Chow group of S) is then a pullback of a corresponding class in the universal case $\lambda_g := c_g(\mathbb{E})$. The Hodge bundle can be extended to a locally free sheaf (again denoted by $\mathbb{E}$) on every smooth toroidal compactification $\tilde{\mathscr{A}_g}$ of $\mathscr{A}_g$ of the type constructed in [5, Chapter VI, Section 4] there. By a slight abuse of notation, we continue to use the notation λ_g for its top Chern class.

The class λ_g is defined over $\mathbb{Z}$, and for each fibre $\mathscr{A}_g \otimes k$ with k a field, it gives rise to a class, also denoted λ_g, in the Chow group $\mathrm{CH}^g(\mathscr{A}_g \otimes k)$ and in $\mathrm{CH}^g(\tilde{\mathscr{A}_g} \otimes k)$. It was proved in [8] that λ_g vanishes in the Chow group $\mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}_g)$ with rational coefficients; however, it does not vanish on $\mathscr{A}_g \otimes k$, and in [2] we studied its order as a torsion class. It also does not vanish in the Chow group $\mathrm{CH}^g_{\mathbb{Q}}(\tilde{\mathscr{A}_g} \otimes k)$. Therefore one may ask for an effective cycle representing the class λ_g on a compactification.

There are several compactifications of $\mathscr{A}_g$. We let $\mathscr{A}_g^*$ be the minimal or Satake compactification as defined in [5]. This compactification $\mathscr{A}_g^*$ is a disjoint union

$$\mathscr{A}_g^* = \mathscr{A}_g \sqcup \mathscr{A}_{g-1} \sqcup \cdots \sqcup \mathscr{A}_0.$$

If $\tilde{\mathscr{A}_g}$ is a suitable smooth toroidal compactification as constructed in [5], we have a natural map $q\colon \tilde{\mathscr{A}_g} \to \mathscr{A}_g^*$ to the Satake compactification. The moduli space $\mathscr{A}_g'$ of rank 1 degenerations is by definition the inverse image of $\mathscr{A}_g \sqcup \mathscr{A}_{g-1} \subset \mathscr{A}_g^*$ under the natural map $q\colon \tilde{\mathscr{A}_g} \to \mathscr{A}_g^*$. The important fact is that the space $\mathscr{A}_g'$ does *not* depend on a choice $\tilde{\mathscr{A}_g}$ of (toroidal) compactification of $\mathscr{A}_g$; it is a *canonical* partial compactification on $\mathscr{A}_g$. If we want a full compactification, then there is not really a unique one, but we must make choices (see [15]).

The space $\mathscr{A}_g'$ parametrizes semiabelian varieties with torus rank ≤ 1. We let Δ_g be the irreducible closed locus of $\mathscr{A}_g'$ which parametrizes the semiabelian varieties that are trivial extensions

$$1 \to \mathbb{G}_m \to X \to A \to 0$$

of a principally polarized abelian variety of dimension $g-1$. Under the map q this cycle is mapped to $\mathscr{A}_{g-1}$ in the Satake compactification. We denote by δ_g the cycle class in the sense of the $\mathbb{Q}$-classes, $[\Delta_g]_{\mathbb{Q}}$, of this codimension g cycle in the Chow group with rational coefficients of codimension g cycles on $\mathscr{A}_g'$. Note that for $g > 1$ (resp., $g = 1$), the generic semiabelian variety that is a trivial extension by a rank 1 torus has 4 (resp., 2) automorphisms, so Δ_g is counted with multiplicity $1/4$ (resp., $1/2$). We refer to [11] and [12] for cycle theory on stacks, but see also [13], [16], and [2]. We now can formulate our result.

THEOREM 1.1

In the Chow group $\mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}_g' \otimes k)$ *of codimension g cycles of the moduli stack of rank*

≤ 1 *degenerations* $\mathscr{A}'_g \otimes k$*, we have the formula*

$$\lambda_g = (-1)^g \zeta(1-2g)\, \delta_g,$$

where δ_g *is the* $\mathbb{Q}$*-class of the locus* Δ_g *of semiabelian varieties that are trivial extensions of an abelian variety of dimension* $g-1$ *with* $\mathbb{G}_m$*.*

Recall that $\zeta(1-2g)$ is a rational number and equals $-b_{2g}/2g$ with b_{2g} the $2g$th Bernoulli number.

Example 1.2
We have $12\,\lambda_1 = \delta_1$, $120\,\lambda_2 = \delta_2$, and $252\,\lambda_3 = \delta_3$.

For $g = 2$ and $g = 3$, there is a canonical toroidal compactification $\tilde{\mathscr{A}_g}$ of $\mathscr{A}_g$, the Delaunay-Voronoi compactification. In [8], van der Geer obtained the following formulas for δ_g: in the rational Chow ring of $\tilde{\mathscr{A}_g}$ for $g = 2$ and $g = 3$,

$$\delta_2 = 120\lambda_2 - \sigma_2, \qquad \delta_3 = 252\lambda_3 - 15\lambda_1^2\sigma_1 + 2\lambda_1\sigma_2,$$

where σ_i denotes a certain class of codimension i lying in the boundary. Our formula gives the part that does not depend on the choice of compactification.

Just as in the case of curves, it is possible to introduce the tautological ring for compactified moduli of abelian varieties. One simply takes the subring of the Chow ring generated by the λ_i in a toroidal compactification (for which the Hodge bundle has been given a toroidal extension). This is seen to be independent of the chosen compactification, using [5, Chapter IV, Theorem 1.1] to reduce to the case when one compactification is dominated by another and using the projection formula in that case. Furthermore, its relations are easily specified without making reference to a toroidal compactification (see Section 3). However, as in the case of curves, sometimes natural loci have classes in the tautological ring, and one then wishes to find the corresponding formulas. This is somewhat problematic, particularly when these classes lie in the boundary as it is not even clear that this question is independent of the toroidal compactification. We suggest introducing instead the *tautological module*, which by definition is the pushdown of the tautological ring to the Satake compactification. Note that as the Satake compactification is (highly) singular, the tautological module is only a subspace of the (rational) Chow homology group (and not the Chow cohomology ring). We end this paper by giving some examples of how to express the classes of natural loci as elements of the tautological module.

2. The proof of Theorem 1.1

In order to prove Theorem 1.1, we may work on a level cover of the moduli space $\mathscr{A}'_g$ for some level $n \geq 3$ prime to the characteristic of the field k and prove the corresponding relation $\lambda_g = (-1)^g \zeta(1-2g)\, n\, \delta_g^{(n)}$ there. Here $\delta_g^{(n)}$ denotes the locus of semiabelian varieties with level n structure which are trivial extensions of a $(g-1)$-dimensional abelian variety by a rank 1 algebraic torus. This has the advantage that we can avoid the problems due to the existence of automorphisms. In the proof we then have to employ an index (n) for all objects. Having said that, we carry out the computation by formally working in level 1 and assuming that the reader knows how to interpret our identities.

In the computation we need a description of the space $\mathscr{A}'_g$. We assume that the reader is familiar with the construction of toroidal compactifications of $\mathscr{A}_g$. It might help the reader to have a look at Mumford's paper [15], where the moduli space of rank 1 degenerations is used. Using the natural map $q : \mathscr{A}'_g \to \mathscr{A}^*_g$, an étale cover of the divisor $B_g := \mathscr{A}'_g \backslash \mathscr{A}_g$ can be identified with the dual of the universal family $\hat{\mathscr{X}}_{g-1} \to \mathscr{A}_{g-1}$, and using the principal polarization, it can be identified with the universal family $\mathscr{X}_{g-1} \to \mathscr{A}_{g-1}$. The cycle Δ_g has as support the image of the zero section $s : \mathscr{A}_{g-1} \to \hat{\mathscr{X}}_{g-1}$.

We recall how a point of $\hat{\mathscr{X}}_{g-1}$ determines a semiabelian variety. If Z is a principally polarized abelian variety of dimension $g-1$ with theta divisor Ξ, then the dual abelian variety $\hat{Z}$ classifies semiabelian varieties that are extensions of group schemes

$$1 \to \mathbb{G}_m \to G \to Z \to 0$$

of Z by $\mathbb{G}_m$. Since the polarization defines an isomorphism $Z \to \hat{Z}$, we can associate a semiabelian variety to a point $z \in Z$. We may view this $\mathbb{G}_m$-extension as a $\mathbb{G}_m$-bundle over Z, and we can take the corresponding $\mathbb{P}^1$-bundle $\rho : \tilde{G} \to Z$. We now glue the zero section $\tilde{G}_0$ and the ∞-section $\tilde{G}_\infty$ over a translation by z to get a nonnormal variety $\bar{G}$. Then $O(\tilde{G}_\infty + \rho^{-1}(\Xi))$ descends to a line bundle L on $\bar{G}$ with $h^0(L) = 1$. In this way we find a compactified semiabelian variety canonically associated to the pair $((Z, \Xi), z)$.

By doing this globally, we see that the moduli stack $\mathscr{A}'_g$ comes with a universal semiabelian variety $\pi' : \mathscr{X}'_g \to \mathscr{A}'_g$ and a relative compactification $\bar{\pi}' : \bar{\mathscr{X}}'_g \to \mathscr{A}'_g$ (i.e., this map $\bar{\pi}'$ is proper). One way to describe it is by taking a smooth compactification $\pi : \tilde{\mathscr{X}}_g \to \tilde{\mathscr{A}}_g$ as constructed in [5] and restricting to $\pi^{-1}(\mathscr{A}'_g)$. The result then does not depend on the choice of $\tilde{\mathscr{X}}_g$ (see [15]). The universal semiabelian variety G over the étale cover $\hat{\mathscr{X}}_{g-1}$ of $B_g \subset \mathscr{A}'_g$ is the $\mathbb{G}_m$-bundle obtained from the Poincaré bundle $P \to \mathscr{X}_{g-1} \times \hat{\mathscr{X}}_{g-1}$ by deleting the zero section. We have the maps

$$G = P - \{(0)\} \longrightarrow \mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1} \xrightarrow{q} \mathscr{A}_{g-1}.$$

The fibre over $x \in B_g$ in the compactification $\bar{\mathscr{X}}'_g$ of the universal family $\mathscr{X}'_g$ of semiabelian varieties is a compactification $\bar{G}$ of a $\mathbb{G}_m$-bundle G over an abelian variety X_{g-1} of dimension $g-1$ as constructed above. (In level $n \geq 3$, it is a compactification of a $(\mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z})$-bundle over an abelian variety X_{g-1} of dimension $g-1$ and is a family of n-cycles of $\mathbb{P}^1$'s over X_{g-1}.) The points where $\bar{\pi}'$ is not smooth are exactly the points of $\bar{G}-G$. So, globally, the locus where $\bar{\pi}'$ is not smooth is the codimension 2 cycle D in $\bar{\mathscr{X}}_g'$ obtained from gluing by a shift the zero section and the ∞-section of the $\mathbb{P}^1$-bundle associated to the Poincaré bundle P over $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$. We may identify an étale cover of the support of D with $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$.

Our proof of Theorem 1.1 is based on an application of the Grothendieck-Riemann-Roch theorem (GRR) to the structure sheaf on the universal semiabelian variety over $\mathscr{A}_g'$. We start with a calculation on a smooth compactification $\tilde{\mathscr{X}}_g$, as constructed in [5], of the universal semiabelian variety. We let $\pi : \tilde{\mathscr{X}}_g \to \tilde{\mathscr{A}}_g$ be the natural morphism; if we restrict π to $\mathscr{A}_g'$, we get $\bar{\pi}' : \bar{\mathscr{X}}_g' \to \mathscr{A}_g'$.

Applying GRR to the structure sheaf $O_{\tilde{\mathscr{X}}_g}$ gives, in the Chow rings with rational coefficients,

$$\mathrm{ch}(\pi_! O_{\tilde{\mathscr{X}}_g}) = \pi_*\big(e^{\mathrm{ch}(O_{\tilde{\mathscr{X}}_g})}\,\mathrm{Td}^\vee(\Omega^1_{\tilde{\mathscr{X}}_g/\tilde{\mathscr{A}}_g})\big) = \pi_*\big(\mathrm{Td}^\vee(\Omega^1_{\tilde{\mathscr{X}}_g/\tilde{\mathscr{A}}_g})\big).$$

Here $\mathrm{Td}^\vee$ is the Todd class (which for a line bundle L equals $c_1(L)/(e^{c_1(L)}-1)$). The relative cotangent sheaf fits in an exact sequence

$$0 \to \Omega^1_{\tilde{\mathscr{X}}_g/\tilde{\mathscr{A}}_g} \to \pi^*(\mathbb{E}) \to \mathscr{F} \to 0$$

with $\mathbb{E}$ the Hodge bundle on $\tilde{\mathscr{A}}_g$ and $\mathscr{F}$ a sheaf with support, where π is not smooth. Note that by [5, Chapter VI, Theorem 1.1], we have

$$\pi^*(\mathbb{E}) = \Omega^1_{\tilde{\mathscr{X}}_g}(\log)/\pi^*\big(\Omega^1_{\tilde{\mathscr{A}}_g}(\log)\big),$$

where log refers to logarithmic poles along the divisors at infinity of $\tilde{\mathscr{X}}_g$ and $\tilde{\mathscr{A}}_g$.

Substituting this in the Riemann-Roch formula, we get

$$\mathrm{ch}\big(\pi_!(O_{\tilde{\mathscr{X}}_g})\big) = \pi_*(F)\mathrm{Td}^\vee(\mathbb{E})$$

with $F := \mathrm{Td}^\vee(\mathscr{F})^{-1}$. Since the cohomology of an abelian variety is the exterior algebra on H^1, the derived sheaf $\pi_!(O_{\tilde{\mathscr{X}}_g})$ equals $\bigwedge^* \mathbb{E}^\vee = \sum_{i=0}^{g}(-1)^i \wedge^i \mathbb{E}^\vee$. By the Borel-Serre formula [1, Lemma 18], we have $\mathrm{ch}\big(\bigwedge^* \mathbb{E}^\vee\big) = \lambda_g \mathrm{Td}(\mathbb{E})^{-1}$. Comparing the terms of degree $\leq g$ in the resulting identity

$$\lambda_g \mathrm{Td}(\mathbb{E})^{-1} = \pi_*(F)\mathrm{Td}^\vee(\mathbb{E})$$

yields the following result of [8].

PROPOSITION 2.1
We have $\pi_*(\mathrm{Td}^\vee(\mathscr{F})^{-1}) = \pi_*(F) = \lambda_g$.

We now restrict to $\bar{\mathscr{X}}_g'$ and $\mathscr{A}_g'$. The sheaf $\mathscr{F}$ has support on D. If u is a fibre coordinate on the $\mathbb{G}_m$-bundle over the abelian scheme $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$ over $\hat{\mathscr{X}}_{g-1}$, then a section of the pullback $\pi^*(\mathbb{E})$ of the Hodge bundle is given by du/u. We now pull the section back to the $\mathbb{P}^1$-bundle and take the residue along the zero section and the ∞-section. This gives an isomorphism of sheaves on $\mathscr{A}_g'$,

$$\mathscr{F} \cong O_{\tilde{D}},$$

where $\tilde{D}$ is the double étale cover of D corresponding to choosing the branches zero and ∞ in the $\mathbb{P}^1$-bundle.

The normal bundle to an étale cover of D given by $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$ is then $N = P \oplus \tau^*(P^{-1})$ with P the Poincaré bundle and τ the map from $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$ to itself defining the translation by which we glue the zero section and the ∞-section of the $\mathbb{P}^1$-bundle corresponding to the Poincaré bundle. On points, τ is given by $\tau(x,\hat{x}) = (x+\hat{x},\hat{x})$. (We identify $\hat{\mathscr{X}}$ with $\mathscr{X}$ if needed.) We write $\alpha_1 = c_1(P)$ and $\alpha_2 = c_1(\tau^*(P^{-1}))$ for the first Chern classes. On the space of rank ≤ 1 degenerations $\bar{\mathscr{X}}_g'$, we then can write $[D] = \alpha_1\alpha_2$.

Let $i\colon D \to \bar{\mathscr{X}}_g'$ be the inclusion. Then if we write

$$\mathrm{Td}^\vee(L) = \frac{c_1(L)}{e^{c_1(L)}-1} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\big(c_1(L)\big)^k$$

with b_k the kth Bernoulli number, we have (cf. Mumford [15, page 303])

$$\bar{\pi}_*'\big(\mathrm{Td}^\vee(O_D)^{-1} - 1\big) = \bar{\pi}_*'\bigg(\sum_{k=1}^{\infty} \frac{(-1)^k b_{2k}}{(2k)!} i_*\Big(\frac{\alpha_1^{2k-1}+\alpha_2^{2k-1}}{\alpha_1+\alpha_2}\Big)\bigg). \tag{1}$$

Consider now the Poincaré bundle P on $X \times \hat{X}$ for an abelian variety X of dimension $g-1$ and dual abelian variety $\hat{X}$. We write p and $\hat{p}$ for the projections on X and $\hat{X}$. If T is a line bundle on $X \cong \hat{X}$ which represents (locally in the étale topology) the principal polarization of $\hat{X}_{g-1}$, then $P = m^*T \otimes p^*T^{-1} \otimes \hat{p}^*T^{-1}$. We find

$$\begin{aligned}\tau^*(P^{-1}) &= \tau^*\big((m^*T)^{-1}\big) \otimes \tau^*p^*T \otimes \tau^*\hat{p}^*T \\ &= \tau^*\big((m^*T)^{-1}\big) \otimes m^*T \otimes \hat{p}^*T\end{aligned}$$

since $p\tau = m$ and $\hat{p}\tau = \hat{p}$. We get

$$P \otimes \tau^*(P^{-1}) \cong \tau^*(m^*T)^{-1} \otimes (m^*T)^{\otimes 2} \otimes p^*T^{-1}.$$

Restriction to a fibre $X \times \hat{x}$ gives

$$t_{-2\hat{x}}(T^{-1}) \otimes (t_x^* T)^{\otimes 2} \otimes T^{-1},$$

and by the theorem of the square, this is trivial on such a fibre $X \times \hat{x}$. This implies that on $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$, we have

$$c_1(N) = c_1\big(P \otimes \tau^*(P^{-1})\big) = \alpha_1 + \alpha_2 = \hat{p}^*(\beta)$$

with β a codimension 1 class on $\hat{\mathscr{X}}_{g-1}$. In order to determine β, we may restrict to the other fibre $0 \times \hat{X}$. Then $P_{|0\times\hat{X}}$ is trivial and $\tau^*(P^{-1})|0 \times \hat{X}$ is the pullback of P^{-1} from the diagonal. But assuming, as we may, that T is symmetric, we find that P restricted to the diagonal is $T^{\otimes 2}$. So as a result, we find on $X \times \hat{X}$ that $\beta = -2c_1(T)$ on $\hat{X}$, and we get an identity on $X \times \hat{X}$,

$$N = P \oplus P^{-1} \otimes \hat{p}^*(T^{-2}). \tag{2}$$

We can consider this as a global identity on D by taking it as a definition of the line bundle $\hat{p}^*(T)$ on $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$. The line bundle T restricts in each fibre $\hat{X}$ of p to $O(\Theta)$ with Θ the theta divisor. Developing the terms in (1), we get expressions of the form

$$\begin{aligned}\bar{\pi}'_*\big(i_*(\alpha_1 + \alpha_2)^r(\alpha_1\alpha_2)^s\big) &= \bar{\pi}'_*\big(i_*(\hat{p}^*(\beta^r))(\alpha_1\alpha_2)^s\big)\\ &= j_*\big(\beta^r\phi_*(D^s)\big),\end{aligned}$$

where ϕ is the restriction to the boundary $\bar{\mathscr{X}}'_g - \mathscr{X}'_g$ of $\bar{\pi}'$ and $j\colon B_g \to \mathscr{A}'_g$ is the inclusion of the boundary of $\mathscr{A}'_g$. Moreover, we use $\bar{\pi}'i = \hat{p}$ and abuse the notation D also for the $\mathbb{Q}$-class of D.

We claim that for dimension reasons the only surviving terms are of the form $j_*(\beta^r)\phi_*(D^{g-1})$. Indeed, the fibres of ϕ have dimension $g-1$. Thus, by Proposition 2.1, the only term in (1) which can contribute to λ_g is

$$\frac{(-1)^g b_{2g}}{(2g)!}\bar{\pi}'_* i_*\big((-1)^{g-1}(2g-1)(\alpha_1\alpha_2)^{g-1}\big).$$

So we need to compute $\bar{\pi}'_* i_*(D^{g-1}) = \pi_*(D^g)$. The identity $\pi_*(\mathrm{Td}^\vee(\mathscr{F})^{-1}) = \lambda_g$ of Proposition 2.1 implies

$$\bar{\pi}'_*(F) = \frac{(-1)b_{2g}}{(2g)!}\bar{\pi}'_* i_*\big((2g-1)(\alpha_1\alpha_2)^{g-1}\big) = \lambda_g.$$

Represent the line bundle P by the divisor Π, and represent the line bundle T by a divisor T (by abuse of notation) on $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$. Then, by (2), we have $\alpha_1 = \Pi$

and $\alpha_2 = -\Pi - 2T$, so

$$(-1)^{g-1}\alpha_1^{g-1}\alpha_2^{g-1} = \Pi^{2g-2} + \sum_{r=1}^{g-1} \binom{g-1}{r} \Pi^{2g-2-r} \hat{p}^*(2T)^r.$$

Now apply GRR to the bundle $P \otimes \hat{p}^*(O(nT))$ on $\mathscr{X}_{g-1} \times_{\mathscr{A}_{g-1}} \hat{\mathscr{X}}_{g-1}$ and the morphism $\hat{p}$. It says

$$\operatorname{ch}\big(\hat{p}_!(P \otimes \hat{p}^* O(nT))\big) = \hat{p}_*(e^{\Pi}) \cdot e^{nT} \cdot \operatorname{Td}^\vee(q^*\mathbb{E}_{g-1}), \tag{3}$$

where $q^*(\mathbb{E}_{g-1})$ is the pullback to $\hat{\mathscr{X}}_{g-1}$ of the Hodge bundle $\mathbb{E}_{g-1}$ on $\mathscr{A}_{g-1}$. But $\hat{p}_!(P \otimes \hat{p}^* O(nT))$ is a sheaf with support (in codimension $g-1$) over the zero section S_0. By applying GRR once again, this time to the inclusion $S_0 \to \hat{\mathscr{X}}_{g-1}$ (cf. [14, page 65]), we see that by viewing

$$\hat{p}_!\big(P \otimes \hat{p}^* O(nT)\big)$$

as a derived sheaf on $\mathscr{X}_{g-1}$, we get $c_i\big(\hat{p}_!(P \otimes \hat{p}^* O(nT))\big) = 0$ for $i < g-1$ and

$$c_{g-1}\big(\hat{p}_!(P \otimes \hat{p}^* O(nT))\big) = (-1)^{g-2}(g-2)!\,[S_0].$$

It then follows, by comparing codimension g classes that are coefficients of the same powers of n on both sides of (3), that

$$\hat{p}_*(\Pi^{2g-2-r})T^r = 0 \quad \text{if } r \neq 0$$

and

$$\hat{p}_*(\Pi^{2g-2-r})T^r = (-1)^{g-1}(2g-2)!\,[S_0] \quad \text{if } r = 0.$$

So we find that

$$\pi_*(D^g) = j_*\big(\hat{p}_*(\Pi^{2g-2})\big) = (-1)^{g-1}(2g-2)![\Delta_g],$$

where Δ_g is the zero section of $\mathscr{X}_{g-1} \to \mathscr{A}_{g-1}$. Interpreting this identity in the right way means taking the $\mathbb{Q}$-class of Δ_g. We thus get

$$\begin{aligned} \lambda_g &= \frac{(-1)b_{2g}}{(2g)!}\bar{\pi}'_* i_*\big((2g-1)(\alpha_1\alpha_2)^{g-1}\big) \\ &= \frac{(-1)b_{2g}}{(2g)!}(2g-1)(2g-2)!(-1)^{g-1}[\Delta_g]_{\mathbb{Q}} \\ &= (-1)^g\zeta(1-2g)[\Delta_g]_{\mathbb{Q}}, \end{aligned}$$

as required. This concludes the proof of Theorem 1.1. □

3. The tautological module

Recall that the Hodge bundle extends to a toroidal compactification $\tilde{A}_g$ as the dual of the Lie algebra of the semiabelian variety that is supposed to exist over $\tilde{A}_g$. Recall also that the subring of $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{A}_g)$ generated by the Chern classes λ_i of this extension is independent of the choice of toroidal compactification. Indeed, the relation

$$(1 + \lambda_1 + \cdots + \lambda_g)\bigl(1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g\bigr) = 1$$

(see [8], [4]) that always exists between the λ_i suffices to show that this ring is a Gorenstein algebra with socle in the top degree, $g(g+1)/2$, and is hence determined by the evaluation map in that degree which is independent of the choice of $\tilde{A}_g$. It is still somewhat unpleasant that this *tautological subring* is a ring of classes living on different toroidal compactifications. We suggest that the pushdown of classes in this ring to the Satake compactification should also be of interest. We begin by showing that it is independent of the choice of toroidal compactification.

DEFINITION-PROPOSITION 3.1
Let $\tilde{A}_g$ be a toroidal compactification with $q \colon \tilde{A}_g \to \mathscr{A}_g^*$ the canonical map to the Satake compactification, and let α be a subset of $\{1, 2, \ldots, g\}$. Then the class $\ell_\alpha := q_*(\lambda_\alpha) \in \mathrm{CH}^{\mathbb{Q}}_{\langle\alpha\rangle}(\mathscr{A}_g^*)$, where $\langle\alpha\rangle := \sum_{i \in \{1,2,\ldots,g\}\setminus\alpha} i$ and $\mathrm{CH}^{\mathbb{Q}}_{\langle\alpha\rangle}(\mathscr{A}_g^*)$ is the Chow homology group tensored with $\mathbb{Q}$, is independent of $\tilde{A}_g$, where $\lambda_\alpha = \prod_{i\in\alpha} \lambda_i$. We call the $\mathbb{Q}$-vector space spanned by these elements the tautological module.

Proof
Any two toroidal compactifications have a common refinement (see [5, pages 97–98, (i), (iii)], so we may assume that one compactification is a refinement of the other. Then the proposition is clear since the λ_i are compatible with pullback, pulling back and pushing down is the identity for a birational map, and pushing down is transitive. □

Of particular interest is, of course, to what extent the cycles of natural subvarieties of $\mathscr{A}_g^*$ lie in the tautological module. The class $\ell_{\{1\}}$ is actually the class of the natural ample line bundle on $\mathscr{A}_g^*$ and hence gives an element of the Chow cohomology group $\mathrm{CH}^1_{\mathbb{Q}}(\mathscr{A}_g^*)$. As such it acts on the Chow homology groups and preserves the tautological module by the projection formula. In particular, the images of the fundamental class under powers of $\ell_{\{1\}}$ lie in the tautological module.

In positive characteristic a less trivial example can be found in [8] and [3]. Consider the closed algebraic subset V_0 of $\mathscr{A}_g \otimes \mathbb{F}_p$ of all abelian varieties with p-rank zero. By Koblitz [10], we know that this is a pure codimension g cycle on $\mathscr{A}_g \otimes \mathbb{F}_p$. It is a complete cycle since abelian varieties of p-rank zero cannot degenerate. Any complete subvariety of $\mathscr{A}_g$ has codimension at least g in $\mathscr{A}_g$ (see [8] and [17]).

THEOREM 3.2 (see [8], [3])
The $\mathbb{Q}$*-cycle class of* V_0 *in* $\mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}^*_g \otimes \mathbb{F}_p)$ *is given by the formula* $[V_0]_{\mathbb{Q}} = (p-1)(p^2-1)\cdots(p^g-1)\,\ell_{\{g\}}$.

Remark 3.3

(i) Recently, Keel and Sadun [9] proved that there is no complete subvariety of codimension g in $\mathscr{A}_g \otimes \mathbb{C}$ for $g \geq 3$.

(ii) In positive characteristic there are many other natural subvarieties whose classes lie in the tautological module.

We now show that our main result can be used to express the top and next to the top boundary components as tautological classes.

THEOREM 3.4

(i) *In the group* $\mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}^*_g)$*, we have* $[\mathscr{A}^*_{g-1}] = ((-1)^g/\zeta(1-2g))\ell_{\{g\}}$.

(ii) *In the group* $\mathrm{CH}^{2g-1}_{\mathbb{Q}}(\mathscr{A}^*_g)$*, we have* $[\mathscr{A}^*_{g-2}] = (1/\zeta(1-2g)\zeta(3-2g))\ell_{\{g-1,g\}}$.

Proof
For part (i), we note that by the excision exact sequence

$$\mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}^*_{g-2}) \to \mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}^*_g) \to \mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}_g \setminus \mathscr{A}^*_{g-2}) \to 0,$$

we may prove it in $\mathscr{A}^*_g \setminus \mathscr{A}^*_{g-2}$. We have a proper map $\mathscr{A}'_g \to \mathscr{A}^*_g \setminus \mathscr{A}^*_{g-2}$, and hence the formula follows from Theorem 1.1 by pushing down the main formula to $\mathscr{A}^*_g$. As for part (ii), on $\mathscr{A}'_g$ we have $\lambda_g\lambda_{g-1} = ((-1)^g/\zeta(1-2g))\lambda_{g-1}\delta_g$, the support of δ_g maps finitely to $\mathscr{A}_{g-1}$, and the restriction of λ_{g-1} to it corresponds to λ_{g-1} on $\mathscr{A}_{g-1}$, which is zero. Hence $\lambda_g\lambda_{g-1}$ is zero on $\mathscr{A}'_g$ and for dimension reasons, using excision, is a multiple of $[\mathscr{A}^*_{g-2}]$. That multiple can be determined by intersecting with $\lambda_1^{(g-1)(g-2)/2}$. Using the intersection numbers in [8, page 72], one sees that

$$\lambda_g\lambda_{g-1}\lambda_1^{(g-1)(g-2)/2}[\tilde{\mathscr{A}}_g] = \Big(\frac{1}{\zeta(1-2g)\zeta(3-2g)}\Big)\lambda_1^{(g-1)(g-2)/2}[\tilde{\mathscr{A}}_{g-2}],$$

and then one uses $\lambda_1^{(g-1)(g-2)/2}[\tilde{\mathscr{A}}_{g-2}] = \lambda_1^{(g-1)(g-2)/2}[\mathscr{A}^*_{g-2}]$. □

The statements in Theorem 3.4 suggest an immediate generalization. To lend some credibility to such a generalization, we sketch a proof of it in positive characteristic.

THEOREM 3.5

In characteristic $p > 0$, we have in $\mathrm{CH}^d_{\mathbb{Q}}(\mathscr{A}_g^*)$ with $d = g(g+1)/2 - (g-i)(g+1-i)/2$ the relation

$$[A^*_{g-i}] = (-1)^i \frac{1}{\prod_{j=1}^{i} \zeta(2j-1-2g)} \ell_{\{g-i+1,\ldots,g\}}.$$

Proof

The idea of the proof is to use the fact that the cycle class of the locus $V_f^{(g)}$ in $\tilde{\mathscr{A}}_g$ of semiabelian varieties of p-rank $\leq f$ is a nonzero multiple of λ_{g-f} (see [8]) and the fact that a semiabelian variety of p-rank $\leq f$ has torus rank $\leq f$.

We use a toroidal compactification $\tilde{\mathscr{A}}_g$ of Faltings-Chai type. The closure $\bar{\Delta}_g$ in $\tilde{\mathscr{A}}_g$ of the locus Δ_g has a cover that is a toroidal compactification $\mathscr{A}^T_{g-1}$ of $\mathscr{A}_{g-1}$ but not necessarily smooth. But there is a smooth toroidal compactification $\tilde{\mathscr{A}}_{g-1}$ and a morphism $\tilde{\mathscr{A}}_{g-1} \to \mathscr{A}^T_{g-1}$ with the property that the pullback of the universal semiabelian variety $\tilde{\mathscr{X}}_g$ to $\tilde{\mathscr{A}}_{g-1}$ is a product of a universal semiabelian variety $\tilde{\mathscr{X}}_{g-1}$ with a torus. Then this can be repeated: the smooth toroidal compactification $\tilde{\mathscr{A}}_{g-1}$ is a compactification of the canonical partial compactification $\mathscr{A}'_{g-1}$, and it contains a locus Δ_{g-1} corresponding to trivial extensions of $(g-2)$-dimensional abelian varieties with $\mathbb{G}_m$. We can then consider the closure $\bar{\Delta}_{g-1}$, and there is a smooth toroidal compactification $\tilde{\mathscr{A}}_{g-2}$ mapping to $\bar{\Delta}_{g-1}$ with a similar property.

From Theorem 1.1, we know that we have a relation in $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{\mathscr{A}}_g)$,

$$\lambda_g \doteq [\bar{\Delta}_g] + \sigma_g,$$

where σ_g is a class with support on $q^{-1}(\mathscr{A}^*_{g-2})$ and where $\doteq$ means equality up to a nonzero multiplicative factor which is a rational number. The relation $[V_1^{(g)}] \doteq \lambda_{g-1}$ in $\mathrm{CH}^{g-1}_{\mathbb{Q}}(\tilde{\mathscr{A}}_g)$ gives

$$\lambda_g \lambda_{g-1} \doteq [V_1^{(g)}] \cdot [\bar{\Delta}_g] + [V_1^{(g)}] \cdot \sigma \doteq [V_1^{(g)}] \cdot [\bar{\Delta}_g]$$

since $V_1^{(g)}$ does not intersect the torus rank ≥ 2 locus $q^{-1}(\mathscr{A}^*_{g-2})$. Now the pullback of the intersection of $V_1^{(g)}$ with $\bar{\Delta}_g$ to $\tilde{\mathscr{A}}_{g-1}$ is exactly the p-rank zero locus $V_0^{(g-1)}$ on $\mathscr{A}_{g-1}$. But we know that

$$[V_0^{(g-1)}] \doteq \lambda_{g-1} \doteq \delta_{g-1}$$

on $\mathscr{A}'_{g-1}$; hence, on the cover $\tilde{\mathscr{A}}_{g-1}$ of $\bar{\Delta}_g$, we get the relation

$$[V_1^{(g-1)}] \cdot \bar{\Delta}_g \doteq \delta_{g-1} + \sigma_{g-1}$$

with σ_{g-1} a class with support on $q^{-1}(\mathscr{A}^*_{g-3})$. Now we use the relation $\lambda_{g-2} \doteq [V_2^{(g)}]$ to get

$$\begin{aligned}\lambda_g\lambda_{g-1}\lambda_{g-2} &\doteq [V_2^{(g)}] \cdot \bar{\Delta}_{g-1} + [V_2^{(g)}] \cdot \sigma_{g-1} \\ &\doteq [V_2^{(g)}] \cdot \bar{\Delta}_{g-1},\end{aligned}$$

and the pullback of the intersection of $V_2^{(g)}$ with $\bar{\Delta}_{g-1}$ to $\tilde{\mathscr{A}}_{g-2}$ is exactly $V_0^{(g-2)}$. But using again a relation $[V_0^{(g-2)}] \doteq \lambda_{g-2} \doteq \delta_{g-1}$ on $\mathscr{A}'_{g-2}$, we see that a nonzero multiple $\lambda_g\lambda_{g-1}\lambda_{g-2}$ is represented by the cycle δ_{g-1} on $\tilde{\mathscr{A}}_g - q^{-1}(\mathscr{A}^*_{g-3})$. Arguing as in the proof of Theorem 3.4, we see that $q_*(\lambda_g\lambda_{g-1}\lambda_{g-2})$ is a nonzero multiple of $[A^*_{g-2}]$. To determine the multiple, we intersect again with the appropriate power of λ_1 (cf. [8]). Proceeding in this way by induction, one deduces the theorem. □

By a simple argument, we can show that another class is also in the tautological module.

PROPOSITION 3.6

The cycle class $[B^*_g]$ of the boundary is the same in the Chow group $\mathrm{CH}^g_{\mathbb{Q}}(\mathscr{A}^*_g)$ as a multiple of the $\mathbb{Q}$-class of the locus of products $X \times E$ of principally polarized abelian varieties of dimension $g-1$ with a fixed elliptic curve E.

Proof

For $g = 1, 2$, see [7]. Consider (for $g > 2$) the space $\mathscr{A}_{g-1,1}$ of products of a principally polarized abelian variety of dimension $g-1$ and an elliptic curve. It is the image of $\mathscr{A}_{g-1} \times \mathscr{A}_1$ in $\mathscr{A}_g$ under a morphism to $\mathscr{A}_g$ which can be extended to a morphism $\mathscr{A}^*_{g-1} \times \mathscr{A}^*_1 \to \mathscr{A}^*_g$. Since an étale cover of $\mathscr{A}_1$ is the affine j-line, we find a rational equivalence between the cycle class of a fibre $\mathscr{A}^*_{g-1} \times \{j\}$ with j a fixed point on the j-line and a multiple of the fundamental class of the boundary B^*_g. □

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