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The Euler characteristic of local systems on the moduli of genus 3 hyperelliptic curves

Gilberto Bini · Gerard van der Geer

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Abstract. For a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\}$ of non-negative integers, we calculate the Euler characteristic of the local system $\mathbb{V}_\lambda$ on the moduli space of genus 3 hyperelliptic curves using a suitable stratification. For some λ of low degree, we make a guess for the motivic Euler characteristic of $\mathbb{V}_\lambda$ using counting curves over finite fields.

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1. Introduction

Let $\mathcal{H}_3$ be the moduli space of genus 3 hyperelliptic curves. It is a 5-dimensional substack of the Deligne-Mumford stack $\mathcal{M}_3$ of smooth curves of genus 3. The universal curve $\pi : \mathcal{M}_{3,1} \to \mathcal{M}_3$ defines a natural local system $R^1\pi_*(\mathbb{Q})$ of rank 6 on $\mathcal{M}_3$. It comes with a non-degenerate symplectic pairing. The inclusion morphism $\iota : \mathcal{H}_3 \to \mathcal{M}_3$ defines a natural local system $\mathbb{V} := \iota^*(R^1\pi_*(\mathbb{Q}))$ on $\mathcal{H}_3$.

For any partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\}$ of weight $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$, consider the irreducible representation of $\mathrm{Sp}(6, \mathbb{Q})$ associated with λ. Any such representation yields a symplectic local system $\mathbb{V}_\lambda$ on $\mathcal{H}_3$, which appears 'for the first time' in the decomposition of

$$\mathrm{Sym}^{\lambda_1-\lambda_2}\mathbb{V} \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}\left(\wedge^2\mathbb{V}\right) \otimes \mathrm{Sym}^{\lambda_3}\left(\wedge^3\mathbb{V}\right).$$

If, for example, $\lambda = \{\lambda_1 \geq 0 \geq 0\}$, then $\mathbb{V}_\lambda = \mathrm{Sym}^{\lambda_1}(\mathbb{V})$.

The cohomology with compact support of $\mathcal{H}_3$ with local coefficients in $\mathbb{V}_\lambda$ is supposed to give interesting motives related to automorphic forms, cf. [3,4,7]. As a first step in understanding this cohomology one wants to know the Euler characteristic of $\mathbb{V}_\lambda$. This was calculated for genus 2 by Getzler in [6]. In the present paper we calculate the Euler characteristic

G. Bini
Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano, Italia (e-mail: Gilberto.Bini@mat.unimi.it)

G. van der Geer
Korteweg-de Vries Instituut, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands (e-mail: geer@science.uva.nl)

$$e_c(\mathcal{H}_3, \mathbb{V}_\lambda) = \sum_{i=0}^{10} (-1)^i \dim H_c^i(\mathcal{H}_3, \mathbb{V}_\lambda)$$

for any local system $\mathbb{V}_\lambda$ on $\mathcal{H}_3$. We do this by using a stratification of $\mathcal{H}_3 \otimes \mathbb{C}$ by a union of quasi-projective varieties $\Sigma(G)$, where G is a finite subgroup of $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$, which acts on $\mathbb{V}_\lambda$. By standard properties of the Euler characteristic of local systems (see, e.g., [2], Theorem 5.13), we thus have

$$e_c(\mathcal{H}_3, \mathbb{V}_\lambda) = \sum_G e_c(\Sigma(G)) \dim(\mathbb{V}_\lambda^G),$$

where $e_c(\Sigma(G))$ is the topological Euler characteristic of $\Sigma(G)$ and $\mathbb{V}_\lambda^G$ is the space of G-invariants. We determine $e_c(\Sigma(G))$ via elementary topological arguments and $\dim(\mathbb{V}_\lambda^G)$ via character theory. Getzler wrote down the generating series of Euler characteristics in [6]. Since for genus 2 already the generating series are unwieldy rational functions, we change tactics and give instead a short algorithm that calculates these numbers efficiently.

This calculation is a step in the program to understand the motivic Euler characteristic

$$\sum_{i=0}^{10} (-1)^i [H_c^i(\mathcal{H}_3, \mathbb{V}_\lambda)],$$

where $[H_c^i(\mathcal{H}_3, \mathbb{V}_\lambda)]$ is the class of the cohomology with compact support in the Grothendieck ring of mixed $\mathbb{Q}$-Hodge structures. The hope is that in analogy to the genus 2 case (cf. [4]), one could use this motivic Euler characteristic to describe properties of Siegel modular forms of genus 3, of which very little is known. In Sect. 5, we provide some conjectural formulas of the motivic Euler characteristic for specific low values of $|\lambda|$ based on calculations over finite fields.

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Throughout the paper, ε_n denotes a primitive n-th root of unity.

2. Stabilizers of hyperelliptic curves

Let C be a hyperelliptic curve of genus 3 over the field of complex numbers $\mathbb{C}$. Then C is a degree two cover of $\mathbb{P}^1$ with eight ramification points. It can be given as a curve in the (X, Y)-plane by an equation of the form $Y^2 = f(X)$, where $f(X)$ is a polynomial in $\mathbb{C}[X]$ of degree 7 or 8.

The group $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ acts on the (X, Y)-plane as follows. An element

$$(A, \xi) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \xi \right) \in \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$$

acts via

$$(A, \xi) \cdot (X, Y) := \left(\frac{aX + b}{cX + d}, \frac{\xi Y}{(cX + d)^4} \right).$$

Suppose that $G \leq \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ stabilizes C. Consider the image G' of G under the projection of $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ onto $SL(2, \mathbb{C})$. Clearly, G' acts as a group of rational transformations on the complex projective line. It also permutes the set of ramification points of C. Note that the kernel of this action is the subgroup generated by the central element $-I$. By the possible actions of G' and the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ (see [8]), one concludes by looking at the size of the orbits that G' must be isomorphic to one of the following groups:

i) the cyclic group C_n of order $n = 2, 4, 14$;
ii) the quaternionic group Q_{4n} of order $4n = 8, 12, 16, 24, 32$;
iii) the group O of symmetries of a cube.

For the purposes of what follows, we briefly recall the presentation of the groups in i), ii), iii) as subgroups of $\mathrm{SL}(2, \mathbb{C})$. Any cyclic group of order n in $\mathrm{SL}(2, \mathbb{C})$ is conjugated to the group generated by the matrix

$$\begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^{-1} \end{pmatrix}.$$

Any quaternionic subgroup of order $4n$, $n \geq 2$, is conjugated to the group with generators

$$S = \begin{pmatrix} \varepsilon_{2n} & 0 \\ 0 & \varepsilon_{2n}^{-1} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Finally, the group O is conjugated to the group generated by the matrices

$$T = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 & \varepsilon_8 \\ \varepsilon_8^3 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Remarkably, the isomorphism type of G' determines the structure of G up to a small number of choices. Indeed, for any matrix $A \in G'$ there exist two non-zero complex numbers $\pm\xi$ such that

$$\xi^2 Y^2 = (cX + d)^8 f\left(\frac{aX + b}{cX + d} \right), \tag{2.1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The assignment

$$u : G' \to \mathbb{C}^*, \quad A \mapsto \xi^2,$$

is a character of a one-dimensional representation of G' because $u(I) = 1$. Thus, the group $G \leq \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ contains all pairs $(A, \pm u(A))$, where A varies in one of the groups G' listed in i), ii), iii), and u is a one-dimensional character of G' that satisfies (2.1). Hence, $\#G = 2\#G'$.

As a consequence, there are only finitely many non-isomorphic groups G which arise as possible stabilizers of genus 3 hyperelliptic curves. Each of them induces a permutation action on a set of eight points in $\mathbb{P}^1$. From this we can deduce a *normal form* of curves which are stabilized by G. Examples and explicit computations can be found, for instance, in [9]. There, the stabilizers are not described as subgroups of $PSL(2,\mathbb{C}) \times \mathbb{C}^*$. It is however easy to verify a correspondence between the two descriptions.

In Table 1 we list all possible groups in terms of G' and u, the associated normal form and the cycle structure of the action of G'. To this end, we need to review some conventional notation from character theory. In general, we shall denote by $\mathbf{1}$ the trivial character of G'. If G' is the cyclic group of order n, there are $n-1$ nontrivial characters χ^k such that

$$\chi^k\left(\begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^{-1} \end{pmatrix}\right) = \varepsilon_n^k, \quad 1 \le k \le n-1.$$

On the other hand, the quaternionic group Q_{4n} has only three non-trivial characters of one-dimensional representations, namely:

χ	$\chi(S)$	$\chi(U)$
χ_0	1	-1
χ_+	-1	$-i^n$
χ_-	-1	i^n

The group O has a unique non-trivial 1-dimensional character ρ. Finally, a cycle under the action of G' will be denoted by $(1^{m_1}2^{m_2}\cdots i^{m_i}\cdots)$. This means that there are m_j orbits of length j for $j \ge 1$.

Table 1. Groups and associated normal forms

Name	(G',u)	normal form $Y^2 = f(X)$	Cycle
G_1	$(C_2,\mathbf{1})$	$(X^2-1)(X^6+\sum_{i=1}^5 a_i X^{6-i}+1)$	(1^8)
G_2	$(C_4,\mathbf{1})$	$X^8+b_1X^6+b_2X^4+b_3X^2+1$	(2^4)
G_3	$(Q_8,\mathbf{1})$	$(X^4+c_1X^2+1)(X^8+c_2X^4+1)$	(4^2)
G_4	(C_4,χ^2)	$X(X^6+d_1X^4+d_2X^2+1)$	$(1^2 2^3)$
G_5	$(Q_{16},\mathbf{1})$	X^8+fX^4+1	(8^1)
G_6	(Q_8,χ_0)	$(X^4-1)(X^4+lX^2+1)$	$(2^2 4^1)$
G_7	$(Q_{12},\mathbf{1})$	$X(X^6+mX^3+1)$	$(2^1 3^2)$
G_8	(Q_{32},χ_-)	X^8-1	(8^1)
G_9	$(O,\mathbf{1})$	X^8+14X^4+1	(8^1)
G_{10}	(Q_{24},χ_-)	$X(X^6-1)$	$(2^1 6^1)$
G_{11}	(C_{14},χ^6)	X^7-1	$(1^1 7^1)$

We remark that the normal forms in Table 1 are equivalent to the equations given in [9], Table 3. For example, the map

$$(X,Y) \mapsto \left(\frac{-iX+i}{X+1}, \frac{\sqrt{8}\,\varepsilon_8}{\sqrt{2-l}(X+1)^4}\right), \quad l \neq 2,$$

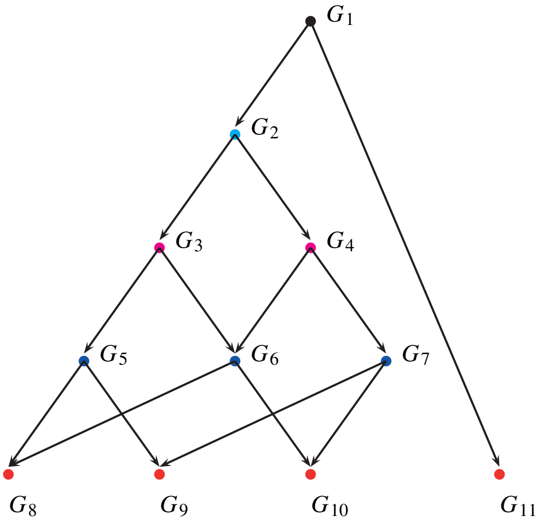
transforms the normal form associated with G_6 to

$$Y^2 = X(X^2-1)(X^4+LX^2+1),$$

where $L = -(12+2l)/(2-l)$. Additionally, the character u changes too. However, this does not affect the calculation of $e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$ - see Sect. 4.

3. The stratification of $\mathcal{H}_3$

For each group G_i in Table 1, define Σ_i to be the locally closed sublocus of $\mathcal{H}_3$ that contains all curves C whose stabilizer is *exactly* G_i. As seen in Sect. 2, the corresponding group G'_i is a permutation group on a set of eight elements. Therefore, the decomposition by automorphism groups described above gives a stratification of $\mathcal{H}_3$, provided that the relation $G'_i \leq G'_j$ is interpreted as an inclusion of permutation groups. In other words, G'_i is a subgroup of G'_j, and any set of eight elements, which is permuted by G'_i, can be decomposed in G'_j-orbits. All possible relations are displayed in the diagram below.



From this diagram, to be justified later, we also deduce information on the strata Σ_i. Actually, we have:

(1) $\mathcal{H}_3 = \bigcup_{i=1}^{11} \Sigma_i$;
(2) $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$;
(3) $\Sigma_j \subseteq \overline{\Sigma}_i$ whenever $G'_i \leq G'_j$.

Note also that the normal form corresponding to each stratum shows that all the Σ_i's are irreducible quasi-projective varieties. As explained in Sect. 1, we need to calculate the topological Euler characteristic e_c of all the strata. Since $e_c(\mathcal{H}_3) = 1$, we work out $e(\Sigma_i)$, $i = 2, \dots, 10$ and we deduce $e_c(\Sigma_1)$.

0-dimensional strata. The stratum Σ_i for $i = 8, 9, 10, 11$ is clearly 0-dimensional and irreducible, so its Euler number is 1.

1-dimensional strata. The strata corresponding to G_5, G_6, G_7 are 1-dimensional. Moreover, let us consider the following subsets of $\mathbb{P}^1$:

$$\mathcal{O}_1 := \{\varepsilon_8^k;\ 0 \leq k \leq 7\},$$
$$\mathcal{O}_2 := \{0, \infty, \pm 1, \pm\varepsilon_3, \pm\varepsilon_3^2\},$$
$$\mathcal{O}_3 := \{\pm\alpha_1, \pm i\alpha_1, \pm 1/\alpha_1, \pm i/\alpha_1\},$$

where α_1 is a root of the polynomial $X^2 - (i+1)X - i$.

It is easy to verify that $\mathcal{O}_1$ is a G'_8-orbit, a union of two G'_5-orbits and a union of three G'_6-orbits. On the other hand, $\mathcal{O}_2$ is a union of three G'_7-orbits, a union of three G'_6-orbits and a union of two G'_{10}-orbits. Finally, $\mathcal{O}_3$ is a full G'_9-orbit, a union of two G'_5-orbits and a union of three G'_7-orbits. This justifies the lower row of directed edges in the above diagram.

As for the Euler number e_c, the following holds.

Proposition 3.1. *The topological Euler characteristic of* Σ_i, $i = 5, 6, 7$, *is equal to* -2.

Proof. We just prove the statement for Σ_5, the other cases being similar. For $f \in \mathbb{C} - \{\pm 2\}$, consider the set of hyperelliptic curves C_f with equation $Y^2 = X^8 + fX^4 + 1$. By direct calculation, two such curves are isomorphic if and only if $f_1 = \pm f_2$. (Indeed, the associated binary forms $T^2 + f_i TU + U^2$ are equivalent, hence have the same discriminant $f_i^2 - 4$.) Note that Σ_8 and Σ_9 are the isomorphism classes of C_0 and C_{14}, respectively. Therefore, there exists an isomorphism $\Phi : \Sigma_5 \cup \Sigma_8 \cup \Sigma_9 \to \mathbb{C} - \{4\}$ which maps the equivalence class of C_f to f^2. Accordingly, the topological Euler characteristic of Σ_5 is -2. $\square$

2-dimensional strata. As readily checked from Table 1, the strata corresponding to G_3 and G_4 have dimension two. It is easy to deduce from the ramification sets in $\mathbb{P}^1$ that the following holds:

$$\Sigma_5 \subset \overline{\Sigma}_3, \quad \Sigma_6 \subset \overline{\Sigma}_3,$$
$$\Sigma_6 \subset \overline{\Sigma}_4, \quad \Sigma_7 \subset \overline{\Sigma}_4.$$

On the other hand, note that Σ_5 does not lie in the closure of Σ_4. Equivalently, there is no set $\mathcal{S}$ of eight elements which is both a union of G'_4-orbits and

G'_5-orbits. Indeed, any set $\mathcal{S} \subset \mathbb{P}^1$ has always two orbits of length one under the action of G'_4. Conversely, the permutation action of G'_5 does not have any fixed point.

Proposition 3.2. *The topological Euler characteristic of* Σ_3 *is* 1.

Proof. The group G_3 corresponds to the pair $(G'_3, \mathbf{1})$, where G'_3 is the quaternionic group $Q_4 \cong C_2 \times C_2$. The group G'_3 induces a permutation action on $\mathbb{P}^1$ via the group V_4 generated by the transformations $x \mapsto -x$ and $x \mapsto 1/x$. Denote by $V_4(x)$ the orbit of x under V_4. Note $\#V_4(a) = 4$ unless $a \in \{0, \infty, 1, -1, i, -i\}$.

We recall that the normal form associated with G_3 is

$$Y^2 = f(X) = (X^4 + c_1X^2 + 1)(X^4 + c_2X^2 + 1). \tag{3.1}$$

Moreover, we have

$$\{x : f(x) = 0\} = \{\pm q_1, \pm 1/q_1, \pm q_2, \pm 1/q_2\}, \tag{3.2}$$

for distinct q_1, q_2 such that $\#V_4(q_1) = \#V_4(q_2) = 4$. Note that $c_i = -q_i^2 - 1/q_i^2$ for $i = 1, 2$.

Let $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ be two curves with stabilizer G_3. They are isomorphic if and only if there exists a rational transformation that maps $\{z : f_1(z) = 0\}$ to $\{z : f_2(z) = 0\}$. All such transformations commute with the elements of V_4. Therefore, two curves are isomorphic if and only if there exists an automorphism of $\mathbb{P}^1/V_4$ which preserves the set $E := \{V(0), V(1), V(i)\}$, i.e. the ramification set of $\mathbb{P}^1 \to \mathbb{P}^1/V_4$. Observe that the map $\mathbb{P}^1 \to \mathbb{P}^1/V_4 \cong \mathbb{P}^1$ sends y to $(y^2 + 1/y^2)/2$.

A curve C with equation (3.1) has a larger stabilizer than G_3 if and only if there exists $M \in \mathrm{SL}(2, \mathbb{C})$ - not in G'_3 - which induces a permutation of (3.2) and a permutation of the set $\{0, \infty, 1, -1, i, -i\}$. By direct inspection, there is only one possible M, namely:

$$M = \begin{pmatrix} \varepsilon_8 & 0 \\ 0 & \varepsilon_8^{-1} \end{pmatrix}.$$

In this case, M induces the automorphism $x \mapsto ix$ on $\mathbb{P}^1$ and the automorphism $z \mapsto -z$ on $\mathbb{P}^1/V_4$. Its fixed points on $\mathbb{P}^1/V_4$ are $V(0)$ and $V(\varepsilon_8)$.

Now, it is possible to give an alternative description of Σ_3, which contains all curves whose stabilizer is *exactly* G_3. Denote by Δ the diagonal in $(\mathbb{P}^1/V_4 - E) \times (\mathbb{P}^1/V_4 - E)$. Define a group W_4 of tranformations of $\mathbb{P}^1/V_4 \times \mathbb{P}^1/V_4$ as follows: W_4 is generated by τ, which interchanges both factors and ι, which simultaneously multiplies both factors by i. Note that W_4 is isomorphic to the Klein four group. Therefore, Σ_3 can be parametrized as

$$\left((\mathbb{P}^1/V_4 - E) \times (\mathbb{P}^1/V_4 - E) - \Delta - Z\right)/W_4,$$

where

$$Z := \{(V(a), V(ia) : a \in \left(\mathbb{P}^1/V_4 - E\right) - V(\varepsilon_8)\}.$$

For the Euler number we get:

$$e_c(\Sigma_3) = \frac{1}{4}((-1) \times (-1) - (-1) - (-2)) = 1.$$

□

Proposition 3.3. *The topological Euler characteristic of* Σ_4 *is* 1.

Proof. The group G_4 corresponds to the pair (G_4', χ^2), where G_4' is cyclic of order 2. Now G_4' induces a permutation action on $\mathbb{P}^1$ via the transformation $x \mapsto -x$. Denote by $\sigma(x)$ the orbit of x under such transformation.

We recall that the normal form associated with G_4 is

$$Y^2 = f(X) = X(X^6 + d_1X^4 + d_2X^2 + 1). \tag{3.3}$$

Moreover, we have

$$\{\infty\} \cup \{z : f(z) = 0\} = \{\infty, 0, \pm a, \pm b, \pm c\}$$

for some distinct $a, b, c \in \mathbb{C}^*$. Therefore, any equation of the form (3.3) corresponds to the 5-point set $\{\sigma(0), \sigma(\infty), \sigma(a), \sigma(b), \sigma(c)\}$ on the $\mathbb{P}^1$ which parametrizes the orbits $\{\sigma(x) : x \in \mathbb{P}^1\}$.

Let $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ be two curves with stabilizer G_4. They are isomorphic if and only if there exists a rational transformation that maps $\{\infty\}\cup\{z : f_1(z) = 0\}$ to $\{\infty\}\cup\{z : f_2(z) = 0\}$ and fixes 0 and ∞. Such a transformation commutes with $x \to -x$. Consequently, $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ are isomorphic if and only if the associated 5-point sets are mapped one onto the other by a rational transformation which preserves $\sigma(0)$ and $\sigma(\infty)$ and permutes the other three points. In other words, an isomorphism class of curves with stabilizer G_4 defines an element in $\mathcal{M}_{0,5}/\mathfrak{S}_3$, where $\mathcal{M}_{0,5}$ is the moduli space of rational 5-pointed curves and $\mathfrak{S}_3$ is the symmetric group of degree three. Conversely, any element in $\mathcal{M}_{0,5}/\mathfrak{S}_3$ determines an equivalence class of curves with stabilizer G_4.

Note that elements in $\mathcal{M}_{0,5}/\mathfrak{S}_3$ can be written as $(0, \infty, 1, \sigma(u), \sigma(v))$ for some distinct $u, v \in \mathbb{P}^1 - \{0, \infty, \pm 1\}$. If $\sigma(u)\sigma(v) = 1$, the corresponding curve has a bigger stabilizer, so it is not an element of Σ_4. As a consequence, this stratum can be identified with $\mathcal{M}_{0,5}/\mathfrak{S}_3 - Y$, where Y is the image of

$$X := \{(0, \infty, 1, \sigma(u), 1/\sigma(u))\} \subset \mathcal{M}_{0,5}$$

under the quotient map onto $\mathcal{M}_{0,5}/\mathfrak{S}_3$. Thus, we have

$$e_c(\Sigma_4) = e_c(\mathcal{M}_{0,5}/\mathfrak{S}_3) - e_c(Y) = 1 - e_c(Y)$$

and

$$e_c(X) = 6e_c(Y) - r,$$

where r is the number of ramification points. Note that $e_c(X) = 2 - 4 = -2$ since $\sigma(u) \notin \{\sigma(0), \sigma(i), \sigma(1), \sigma(\infty)\}$. Additionally, $r = 2$ since the quotient map onto $\mathcal{M}_{0,5}/\mathfrak{S}_3$ is ramified over X when $\sigma(u)$ is the orbit of a primitive third root of unity. Hence, the statement is completely proved. □

3-dimensional strata. There is only one 3-dimensional stratum, namely Σ_2. As readily checked, both Σ_3 and Σ_4 lie in the closure of Σ_2.

Proposition 3.4. *The topological Euler characteristic of* Σ_2 *is* 2.

Proof. The group G_2 corresponds to the pair $(G_2', \mathbf{1})$, where G_2' is cyclic of order two. As in Proposition 3.3, G_2' induces a permutation action on $\mathbb{P}^1$ via the transformation $x \mapsto -x$. Again, denote by $\sigma(x)$ the orbit of x under such transformation.

We recall that the normal form associated with G_2 is

$$Y^2 = f(X) = X^8 + b_1X^6 + b_2X^4 + b_3X^2 + 1. \tag{3.4}$$

Moreover, we have

$$\{z : f(z) = 0\} = \{\pm p_1, \pm p_2, \pm p_3 \pm p_4\}$$

for some distinct $p_1, p_2, p_3, p_4 \in \mathbb{C}^*$. Therefore, any equation of the form (3.4) corresponds to the 4-point set $\{\sigma(p_1), \sigma(p_2), \sigma(p_3), \sigma(p_4)\}$ on the $\mathbb{P}^1$ which parametrizes the orbits $\{\sigma(x) : x \in \mathbb{P}^1\}$.

Let $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ be two curves with stabilizer G_2. They are isomorphic if and only if there exists a rational transformation that maps $\{z : f_1(z) = 0\}$ to $\{z : f_2(z) = 0\}$. All transformations preserve the cycle structure of the group G_2'. Consequently, $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ are isomorphic if and only if the associated 4-point sets are mapped one onto the other by a rational transformation. In other words, equivalence of curves with equation (3.4) corresponds to equivalence of 4-tuples of points in $\mathbb{P}^1$ under the action of $\mathrm{SL}(2, \mathbb{C})$ and the symmetric group of degree 4. Thus, an isomorphism class of curves stabilized by G_2 defines a point in $\mathcal{M}_{0,4}/\mathfrak{S}_4$, where $\mathcal{M}_{0,4}$ is the moduli space of 4-pointed rational curves and $\mathfrak{S}_4$ is the symmetric group of order 4. Note that $e_c(\mathcal{M}_{0,4}/\mathfrak{S}_4) = 1$: see, for instance, [1].

We finally observe that Σ_2 is not the whole of $\mathcal{M}_{0,4}/\mathfrak{S}_4$. In fact, we need to disregard all curves with extra automorphisms, i.e., the ones in lower dimensional strata. Therefore.

$$\begin{aligned} e_c(\Sigma_2) &= e_c(\mathcal{M}_{0,4}/\mathfrak{S}_4) - \sum_{i=3}^{10} e_c(\Sigma_i) \\ &= 1 - (-6 + 2 + 3) = 2. \end{aligned}$$

□

Table 2. Some topological invariants of the strata Σ_i.

i	1	2	3	4	5	6	7	8	9	10	11
$\dim(\Sigma_i)$	5	3	2	2	1	1	1	0	0	0	0
$e_c(\Sigma_i)$	−1	2	1	1	−2	−2	−2	1	1	1	1

In Table 2, we list the dimension and the topological Euler characteristic of all the strata in $\mathcal{H}_3$.

4. The calculation of $e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$

Let $\gamma_j : \Sigma_j \to \mathcal{H}_3$ be the embedding of Σ_j in $\mathcal{H}_3$. By the properties of the Euler characteristic of local systems, we have

$$e_c(\mathcal{H}_3, \mathbb{V}_\lambda) = \sum_{j=1}^{11} e_c\left(\Sigma_j, \gamma_j^*(\mathbb{V}_\lambda)\right).$$

On the other hand, $\gamma_j^*(\mathbb{V}_\lambda)$ is a local system on Σ_j with respect to G_j. Hence, (4.1) can be written as

$$e_c(\mathcal{H}_3, \mathbb{V}_\lambda) = \sum_{j=1}^{11} e_c(\Sigma_j)\dim(\mathbb{V}_\lambda^{G_j}),$$

where $\mathbb{V}_\lambda^{G_j}$ is the space of G_j-invariants. In Sect. 3, we computed $e_c(\Sigma_j)$. Now, we work out the dimension of the corresponding invariant subspaces.

By definition, the fibre of the local system $\mathbb{V}_{(1,0,0)}$ over a curve C is given by the cohomology group $H^1(C;\mathbb{Q})$. $\mathbb{V}_\lambda$ is thus obtained from the $\mathrm{Sp}(6,\mathbb{Q})$-module $\mathbb{V}_{(1,0,0)}$ by standard constructions in representation theory (cfr. [5], pp. 262). Obviously, any group G in Table 1 acts on $\mathbb{V}_{(1,0,0)}$. This action yields a homomorphism $\eta : G \to \mathrm{Sp}(6,\mathbb{Q})$. Let (A,ξ) be an element in G, where A is a matrix with eigenvalues a and a^{-1}. By Corollary 3 in [6], the eigenvalues of $\eta(g)$ are given by

$$a^2\xi, \quad a^{-2}\xi^{-1}, \quad a^{-2}\xi, \quad a^2\xi^{-1}, \quad \xi, \quad \xi^{-1}.$$

As a consequence, it is possible to compute the dimension of the G-invariant subspace of $\mathbb{V}_\lambda$ by elementary character theory. More specifically, let J_d be the symmetric function

$$J_d(x_1,x_2,x_3) = h_d(x_1,x_1^{-1},x_2,x_2^{-1},x_3,x_3^{-1}),$$

where h_d is the complete symmetric function of degree d in six variables. Moreover, for any $\{\lambda = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\}$, we denote by J_λ the determinant of the 3×3 matrix whose i-th row is

$$(J_{\lambda_i-i+2}\ J_{\lambda_i-i+2} + J_{\lambda_i-i}\ J_{\lambda_i-i+3} + J_{\lambda_i-i-1}).$$

Table 3. The sets Y_i

Y_1	$(1, 1)$
Y_2	$(1, 1), (i, 1)$
Y_3	$(1, 1), (i, 1)^3$
Y_4	$(1, 1), (i, i)$
Y_5	$(1, 1), (\varepsilon_{16}^2, 1), (\varepsilon_{16}^6, 1), (i, 1)^5$
Y_6	$(1, 1), (i, 1), (i, i)^2$
Y_7	$(1, 1), (\varepsilon_{12}^2, 1), (\varepsilon_{12}^4, 1), (i, 1)^3$
Y_8	$(1, 1), (\varepsilon_{16}, i), (\varepsilon_{16}^2, 1), (\varepsilon_{16}^3, i), (\varepsilon_{16}^5, i), (\varepsilon_{16}^6, 1), (\varepsilon_{16}^7, i), (i, i)^4, (i, 1)^5$
Y_9	$(1, 1), (i, 1)^9, (\varepsilon_{12}^2, 1)^4, (\varepsilon_{12}^4, 1)^4, (\varepsilon_{16}^2, 1)^3, (\varepsilon_{16}^6, 1)^3$
Y_{10}	$(1, 1), (\varepsilon_{14}, \varepsilon_{14}^3), (\varepsilon_{14}^2, \varepsilon_{14}^6), (\varepsilon_{14}^3, \varepsilon_{14}^9), (\varepsilon_{14}^4, \varepsilon_{14}^{12}), (\varepsilon_{14}^5, \varepsilon_{14}), (\varepsilon_{14}^6, \varepsilon_{14}^4)$
Y_{11}	$(1, 1), (i, 1)^9, (\varepsilon_{12}^2, 1)^4, (\varepsilon_{12}^4, 1)^4, (\varepsilon_{16}^2, 1)^3, (\varepsilon_{16}^6, 1)^3$
Y_{11}	$(1, 1), (\varepsilon_{12}, i), (\varepsilon_{12}^5, i), (\varepsilon_{12}^2, 1), (\varepsilon_{12}^4, 1), (i, i)^4, (i, 1)^3$

By Proposition 24.22 in [5], the following holds:

$$\dim\left(\mathbb{V}_\lambda^G\right) = \frac{1}{\#G} \sum_{g \in G} J_\lambda(a^2\xi, a^{-2}\xi, \xi).$$

For each of the groups G_i we can list the pairs (a, ξ) that occur as g runs through G. If (a, ξ) occurs, then $(a, -\xi)$, $(-a, \xi)$ and $(-a, -\xi)$ occur too. For each G_i in Table 3 we give a set Y_i of cardinality $\#G_i/4$ of pairs (a, ξ) with multiplicity (indicated by an exponent). The set Y_i has the following property. If we replace $(a, \xi) \in Y_i$ by the 4 elements $(\pm a, \pm\xi)$ we get all the pairs with multiplicity corresponding to the $g \in G$. This is indicated by the notation $(\pm a, \pm\xi) \in Y_i$.

Theorem 4.1. *The Euler characteristic $e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$ is given by*

$$e_c(\mathcal{H}_3, \mathbb{V}_\lambda) = \sum_{i=1}^{11} \frac{e_c(\Sigma_i)}{\#G_i} \sum_{(\pm a, \pm\xi) \in Y_r} J_\lambda(a^2\xi, a^{-2}\xi, \xi),$$

where the Euler numbers $e_c(\Sigma_i)$ and the sets Y_i are given in Tables 2 and 3.

For example, the elements of the group G_1 are $(\pm \mathrm{I}, \pm 1)$. If $\lambda = (k, 0, 0)$, then the contribution from this group yields

$$\begin{aligned}\dim(\mathbb{V}_{(k,0,0)}^{G_1}) &= \frac{1}{4}\{2h_k(1, 1, 1, 1, 1, 1) + 2h_k(-1, -1, -1, -1, -1, -1)\} \\ &= \frac{1}{2}\binom{k+5}{k}\left(1 + (-1)^k\right).\end{aligned}$$

In the following table we give the values of $e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$ for all λ of weight ≤ 10. Note that because of the hyperelliptic involution $e_c(\mathcal{H}_3, \mathbb{V}_\lambda) = 0$ if the weight is odd. This gives us a first check on the Euler numbers obtained. Also the integrality of the numbers gives a check.

Table 4. Some values of $e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$

$(\lambda_1, \lambda_2, \lambda_3)$	$e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$	$(\lambda_1, \lambda_2, \lambda_3)$	$e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$
(0,0,0)	1	(5,2,1)	−10
(2,0,0)	−1	(4,4,0)	−5
(1,1,0)	0	(4,3,1)	−4
(4,0,0)	−1	(4,2,2)	−7
(3,1,0)	0	(3,3,2)	−2
(2,2,0)	−1	(10,0,0)	−17
(2,1,1)	0	(9,1,0)	−22
(6,0,0)	−5	(8,2,0)	−43
(5,1,0)	−2	(8,1,1)	−8
(4,2,0)	− 5	(7,3,0)	−34
(4,1,1)	0	(7,2,1)	−32
(3,3,0)	0	(6,4,0)	−37
(3,2,1)	0	(6,3,1)	−26
(2,2,2)	−3	(6,2,2)	−27
(8,0,0)	−7	(5,5,0)	−6
(7,1,0)	−8	(5,4,1)	−22
(6,2,0)	−13	(5,3,2)	−12
(6,1,1)	−2	(4,4,2)	−15
(5,3,0)	−10	(4,3,3)	0

5. Some remarks on the motivic Euler characteristic

For partitions of small degree $|\lambda|$ it is not unreasonable to expect that all cohomology of $\mathbb{V}_\lambda$ is Tate, i.e., that the motivic Euler characteristic

$$E_c(\mathcal{H}_3, \mathbb{V}_\lambda) = \sum_{i=0}^{10} (-1)^i [H_c^i(\mathcal{H}_3, \mathbb{V}_\lambda)]$$

is a polynomial in L, the Tate motive of weight 2. It is well known that $E_c(\mathcal{H}_3, \mathbb{V}_0) = L^5$, see [10] for the complex case. One can calculate the trace of Frobenius on the ℓ-adic variant of $\mathbb{V}_\lambda$ in characteristic p on $\mathcal{H}_3 \otimes \mathbb{F}_p$ by summing

$$\sum_C \mathrm{Tr}(F, \mathbb{V}_\lambda(H^1))/\#\mathrm{Aut}_{\mathbb{F}_p}(C),$$

where C runs over a complete set of representatives of the $\mathbb{F}_p$-isomorphism classes of hyperelliptic curves of genus 3 over $\mathbb{F}_p$. We found that the following guesses for the motivic Euler characteristic are compatible with the values of $e_c(\mathcal{H}_3, \mathbb{V}_\lambda)$ and with these traces for $p = 3$ and $p = 5$ and even for $p = 2$.

Table 5. Conjectural motivic Euler characteristics

λ	$E_c(\mathcal{H}_3, \mathbb{V}_\lambda)$
(0, 0, 0)	L^5
(2, 0, 0)	-1
(1, 1, 0)	0
(4, 0, 0)	$L^2 - 2$
(3, 1, 0)	$L^2 - 1$
(2, 2, 0)	$-L^6 + L^2 - 1$
(2, 1, 1)	$L^5 - L^4 - L^3 + L^2$

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