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Exploring Modular Forms and the Cohomology of Local Systems on Moduli Spaces by Counting Points

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Abstract

We report on a joint project in experimental mathematics with Jonas Bergström and Carel Faber where we obtain information about modular forms by counting curves over finite fields.

1 Introduction

According to Arnold mathematics is the part of physics where experiments are cheap. I take it that he meant "cheap" in the literal sense, that is, not requiring large investments of goods and money. But experiments in mathematics may require a considerable investment of time, mental energy and computational power. On the other hand they can be extremely rewarding. Here we report on a project of experimental mathematics concerning moduli spaces and modular forms that has been going on for quite some time. We used counts of points over finite fields to explore the cohomology of local systems on moduli spaces of low genus curves ([24, 7, 8, 11]). As a side result it produces a lot of information about Siegel modular forms of low degree. This comes about since on the one hand modular forms admit a cohomological description as part of the cohomology of local systems on moduli spaces of abelian varieties and on the other hand abelian varieties of low dimension allow a description in terms of curves. By using comparison theorems for the cohomology we can gauge this cohomology by looking at the situation over a finite field and listing all the isomorphism classes of curves of this genus over

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the finite field and the order of their automorphism groups, and by calculating the action of Frobenius on the cohomology of the corresponding curve.

Birch made such counts for elliptic curves in the 1960s in [12] and from his formulas one sees that one gets information about the traces of the Hecke operators on the spaces of cusp forms. In a joint project with Carel Faber we started counting points of the moduli spaces $\mathcal{M}_{2,n}$ of n-pointed curves of genus 2 over finite fields in order to probe the cohomology of these moduli spaces. We realized that knowledge of the cohomology of the moduli spaces $\mathcal{M}_{2,n}$ is equivalent to information about the cohomology of local systems on $\mathcal{M}_2$ and also realized that this way we could get information about vector-valued Siegel modular forms. It turned out to be wonderfully effective and our heuristic results have since then been turned into theorems. In joint work with Bergström we extended this to local systems on the moduli space $\mathcal{M}_2[2]$ of curves of genus 2 with a level 2 structure and then later also to the case of $\mathcal{M}_3$, the moduli of curves of genus 3, where it makes predictions about Siegel modular forms of degree 3. It was a surprise that this was feasible and it thus created a new window on Siegel modular forms. Apart from these cases we also looked at the case of Picard modular forms. Another chapter is the case of modular forms on $\mathcal{M}_3$, called Teichmüller modular forms. We will not treat this last aspect here and have to refer to future papers.

One of the nice aspects of the theory of elliptic modular forms is the availability of concrete examples and accessibility of the topic for doing experiments. Concrete examples for higher degree Siegel modular forms were much harder to get by, especially for vector-valued modular forms. But recently things have changed quite a lot. For example the work of Chenevier-Renard and Taïbi (see [15, 61]) where applying Arthur's results on the trace formula (even if all has not yet been proved) provided a wealth of new data on Siegel modular forms. No doubt we can await many new surprises and discoveries in this beautiful corner of mathematical nature.

Finally, I would like to thank my collaborators Jonas Bergström and Carel Faber for their comments and the continued pleasant cooperation in this project. I also thank Fabien Cléry with whom I worked on the purely modular forms side of the project. Thanks to Dan Petersen for a useful remark.

2 Modular forms

Every mathematician knows or should know the definition of (elliptic) modular form: a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ on the upper half plane $\mathfrak{H}$ of $\mathbb{C}$ satisfying (for a fixed $k \in \mathbb{Z}$)

$$f((a\tau + b)(c\tau + d)^{-1}) = (c\tau + d)^k f(\tau)$$

for all $\tau \in \mathfrak{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, in particular $f(\tau + 1) = f(\tau)$, and thus admitting a Fourier series

$$f = \sum_{n \in \mathbb{Z}} a(n)\, q^n \qquad \text{with } q = e^{2\pi i \tau}\,,$$

where we require additionally that $a(n) = 0$ for $n < 0$. The exponent $k \in \mathbb{Z}$ is called the weight of the modular form. Examples of modular forms can be found everywhere in mathematics, and in fact they appeared early on in our history, for example in the papers of Jacobi, but the notion of a modular form was formalized only at the end of the nineteenth century, apparently by Klein who introduced the word “Modulform”.

Modular forms of given weight k form a finite-dimensional complex vector space denoted by M_k (or $M_k(\mathrm{SL}(2,\mathbb{Z}))$) and since the product of two modular forms of weight k and l is a modular form of weight $k+l$ they form a graded ring and $\mathbb{C}$-algebra

$$M = \bigoplus_k M_k\,.$$

A modular form vanishing at infinity, that is, with $a(0) = 0$, is called a cusp form. The subspace of cusp forms is denoted by S_k. The cusp form of smallest non zero weight is $\Delta = \sum_{n=1}^{\infty} \tau(n)\, q^n$ which can be defined by an infinite product

$$q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24\,q^2 + 252\,q^3 - 1472\,q^4 + 4830\,q^5 + \cdots$$

and this turns out to be a cusp form of weight 12 generating the ideal of cusp forms in the algebra M.

The structure of this $\mathbb{C}$-algebra M is known: it is a polynomial algebra

$$M = \mathbb{C}[E_4, E_6],$$

where the modular forms E_4 and E_6 of weight 4 and 6 are Eisenstein series defined for even $k \geqslant 4$ via

$$E_k = \frac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-k}$$

where (c,d) runs over coprime pairs of integers and we assume $k \geqslant 4$ to get convergence; it has the Fourier series

$$1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

where $\sigma_r(n) = \sum_{1 \leqslant d | n} d^r$ and B_k is the kth Bernoulli number (defined by the series $x/(e^x - 1) = \sum_{k=0}^{\infty} B_k\, x^k/k!$). The cusp form Δ equals the expression $(E_4^3 - E_6^2)/1728$.

We can replace $\mathrm{SL}(2,\mathbb{Z})$ by a subgroup of finite index, like for example the subgroup $\Gamma_0(N)$ where we require for a matrix $(a,b;c,d)$ that c is divisible by N. Then we find more modular forms. However, for the most part of this report we restrict to the case of level $N = 1$.

Modular forms show up unexpectedly in many parts of mathematics, for example in the theory of lattices as generating series for the number of vectors of given norm in an even definite unimodular lattice. They show up ubiquitously if one studies the moduli of elliptic curves or other algebro-geometric objects. But they also show up in mathematical physics. As a rule the Fourier coefficients of these modular forms show interesting arithmetical properties and seem to contain magical information.

It was Hecke who saw around 1937 how to extract this information from the Fourier series. He introduced the operators $T(n)$ for $n \in \mathbb{Z}_{\geqslant 1}$ named after him. These operators form a commutative algebra of operators. Hecke's student Petersson introduced a positive definite hermitian product on the space of cusp forms and the Hecke operators are hermitian for this product. We thus can find a basis of eigenvectors, called eigenforms which are eigenvectors for all the Hecke operators. Then one shows that for such a nonzero eigenform $f = \sum a(n)q^n$ with eigenvalue $\lambda(m)$ under $T(m)$ we have

$$\lambda(m)a(1) = a(m)$$

so that $a(1) \neq 0$ and when we divide f by $a(1)$ to get $a(1) = 1$ we see that then the Fourier coefficients are the eigenvalues.

It was also Hecke who noticed that the fact that a modular form of weight k is an eigenform for the Hecke algebra implies that the formal L-series of f has an Euler product:

$$L(f,s) = \sum_{n \geqslant 1} \frac{a(n)}{n^s} = \prod_p \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}} \,.$$

For example, for the Eisenstein series one finds $L(E_k, s) = \zeta(s)\zeta(s+1-k)$ with $\zeta(s)$ the Riemann zeta function. Hecke also showed that the L-series of a cusp form can be extended analytically to a holomorphic function of s for all $s \in \mathbb{C}$ and satisfies a functional equation, namely

$$\Lambda(f,s) = (-1)^{k/2}\Lambda(f, k-s)$$

with $\Lambda(f,s) = \Gamma(s)(2\pi)^{-s}L(f,s)$.

Let f be a (normalized with $a(1) = 1$) cusp form of weight k that is an eigenform of the Hecke algebra. We denote the totally real number field obtained by adjoining the eigenvalues of f to $\mathbb{Q}$ by K_f. Deligne showed [21] that to such a cusp form one can associate an ℓ-adic representation

$$\rho_{\ell,f} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2, K_f \otimes \mathbb{Q}_\ell)$$

of the Galois group of $\mathbb{Q}$ with the property that for a prime p different from ℓ the image $\rho_{\ell,f}(F_p)$ of a geometric Frobenius element F_p has trace $\lambda(p)$, the eigenvalue of f under the operator $T(p)$, and determinant p^{k-1}. This was conjectured by Serre [55]. This result exhibited the number theoretical meaning of modular forms. The origin of these representations (namely the cohomology of modular varieties) also enabled Deligne to apply his (later) result on the eigenvalues of Frobenius on the cohomology of algebraic varieties to these eigenvalues, see [21]. We shall see in Section 4 where this representation lives.

3 Siegel modular forms

The notion of modular form (on $\mathrm{SL}(2,\mathbb{Z})$) generalizes. Consider a symplectic lattice $\mathbb{Z}^{2g}$, say with basis $e_1,\ldots,e_g,f_1,\ldots,f_g$, which pair by $\langle e_i,e_j\rangle = 0 = \langle f_i,f_j\rangle$ and $\langle e_i,f_j\rangle = \delta_{ij}$, the Kronecker delta. Then we let

$$\Gamma_g = \mathrm{Sp}(2g,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^{2g},\langle\cdot,\cdot\rangle)$$

be the automorphism group of this lattice, the so-called Siegel modular group. For $g=1$ one finds back $\mathrm{SL}(2,\mathbb{Z})$. The action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane generalizes to an action of Γ_g on the Siegel upper half space

$$\mathfrak{H}_g = \{\tau \in \mathrm{Mat}(g\times g,\mathbb{C}) : \tau^t = \tau, \mathrm{Im}(\tau) > 0\}$$

via

$$\tau \mapsto (a\tau+b)(c\tau+d)^{-1}$$

for an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g$$

where the decomposition of the $2g\times 2g$ matrix γ in four $g\times g$ blocks corresponds to the decomposition of $\mathbb{Z}^{2g}$ as a direct sum of two isotropic sublattices generated by the e_i and the f_i. Often we shall write $\gamma = (a,b;c,d)$ for such a matrix.

We can consider for $g>1$ holomorphic functions $f\colon \mathfrak{H}_g \to \mathbb{C}$ satisfying

$$f((a\tau+b)(c\tau+d)^{-1}) = \det(c\tau+d)^k f(\tau)$$

for all $\tau\in\mathfrak{H}_g$ and $\gamma\in\Gamma_g$. This is the notion of a Siegel modular form of degree g (or genus g) and weight k. But there is a wider generalization. Let

$$\rho : \mathrm{GL}(g,\mathbb{C}) \to \mathrm{Aut}(W)$$

be a finite dimensional irreducible complex representation of $\mathrm{GL}(g,\mathbb{C})$ on the vector space W. Then we can consider holomorphic maps

$$f : \mathfrak{H}_g \to W$$

satisfying

$$f((a\tau+b)(c\tau+d)^{-1}) = \rho(c\tau+d)f(\tau)$$

for all $\tau\in\mathfrak{H}_g$ and $(a,b;c,d)\in\Gamma_g$. Such a function f is called a Siegel modular form of degree g and weight ρ. The space of all such modular forms is denoted by $M_\rho(\Gamma_g)$. If $\rho = \det^k$ we retrieve the notion of scalar-valued Siegel modular form of weight k. It is a reflection of the Hartogs extension theorem that for $g>1$ we need not demand a holomorphicity condition like we did for $g=1$ (when we demanded $a(n)=0$ for $n<0$). In fact, we can write the Fourier series of a Siegel modular form f as

$$f(\tau) = \sum_n a(n) q^n$$

where the index n runs over all $g \times g$ symmetric matrices whose entries are half-integral and with integral diagonal entries and where we employ the shorthand

$$q^n = e^{2\pi i \mathrm{Tr}(n\,\tau)}$$

and $a(n)$ is a vector in the vector space W; the Koecher principle says that if $a(n) \neq 0$ then n is positive semi-definite.

There is a way to obtain a Siegel modular form of degree $g-1$ from a form f of degree g by taking a limit

$$\Phi(f)(\tau') = \lim_{t\to\infty} f\begin{pmatrix} \tau' & 0 \\ 0 & i\,t \end{pmatrix} \quad \text{with } t \in \mathbb{R}_{>0} \text{ and } \tau' \in \mathfrak{H}_{g-1}\,.$$

This operator is called the Siegel operator. Forms for which $\Phi(f)$ is zero are called cusp forms.

Like for $g = 1$ we can form a graded $\mathbb{C}$-algebra of scalar-valued Siegel modular forms on Γ_g

$$M(\Gamma_g) = \bigoplus_k M_k(\Gamma_g)$$

where $M_k(\Gamma_g) = M_{\det^k}(\Gamma_g)$. The structure of this algebra is known for $g = 2$ and $g = 3$ only. For $g = 2$ Igusa proved that the subalgebra of forms of even weight is a polynomial algebra given by

$$M^{\mathrm{ev}}(\Gamma_2) = \bigoplus_{k\equiv 0 (\mathrm{mod}\ 2)} M_k(\Gamma_2) = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}]\,.$$

Here E_4 and E_6 are Eisenstein series, now for $g = 2$, defined by series

$$E_k = \sum_{(c,d)} \det(c\tau + d)^{-k}$$

with a sum over non-associated pairs of co-prime symmetric integral matrices (where non-associated is taken with respect to left multiplication by elements of $\mathrm{GL}(2,\mathbb{Z})$), and χ_{10} and χ_{12} are cusp forms; up to a normalization they can be given by $E_{10} - E_4E_6$ and $E_{12} - E_6^2$. The algebra of modular forms of all weights is an extension of $M^{\mathrm{ev}}(\Gamma_2)$ generated by a form χ_{35} of weight 35 that satisfies a quadratic relation $\chi_{35}^2 = P(E_4, E_6, \chi_{10}, \chi_{12})$ expressing it as a polynomial in the even weight modular forms. For $g = 3$ the structure of $M(\Gamma_3)$ was determined by Tsuyumine [64]; it has 34 generators and thus becomes rather complicated to handle.

Vector-valued Siegel modular forms have attracted much less attention than the scalar-valued ones, but as we shall see they are the natural generalization of the elliptic modular forms. Much less is known about vector-valued Siegel modular forms than about scalar-valued forms.

The quotient space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}_1$ is the moduli space of complex elliptic curves; similarly, $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathfrak{H}_g$ is the moduli space of principally polarized complex abelian varieties of dimension g:

$$\mathcal{A}_g(\mathbb{C}) := \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathfrak{H}_g\,.$$

It can be viewed as the fiber over the infinite place of a moduli stack $\mathcal{A}_g$ over $\mathbb{Z}$ parametrizing principally polarized abelian varieties and this comes equipped with a universal abelian variety

$$\pi : \mathcal{X}_g \longrightarrow \mathcal{A}_g$$

with fibre $X_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ over the orbit of τ in $\mathcal{A}_g(\mathbb{C})$. We should view these moduli spaces (or quotients) as stacks or orbifolds.

Modular forms can be interpreted as sections of a vector bundle: Γ_g acts on $\mathfrak{H}_g \times \mathbb{C}^g$ by

$$(\tau, z) \mapsto (\gamma\tau, (c\tau + d)z)$$

where $\gamma = (a, b; c, d)$. This defines a vector bundle (in the orbifold sense). More intrinsically, we have

$$\mathbb{E}_g = \pi_*\Omega^1_{\mathcal{X}_g/\mathcal{A}_g},$$

the Hodge bundle, and its restriction to $\mathcal{A}_g(\mathbb{C})$ corresponds to the standard representation of $\mathrm{GL}(g, \mathbb{C})$. For every irreducible representation ρ of $\mathrm{GL}(g, \mathbb{C})$ we have a corresponding bundle U_ρ on $\mathcal{A}_g(\mathbb{C})$ produced by applying an appropriate Schur functor to $\mathbb{E}$; or more concretely, it can be described as the quotient of the action of Γ_g on $\mathfrak{H}_g \times W$ with W the representation space of ρ by $(\tau, w) \mapsto (\gamma\tau, \rho(c\tau+d)w)$. Then we can interpret modular forms as sections: for $g \geqslant 2$ we have

$$M_\rho(\Gamma_g) = \text{space of sections of } U_\rho \text{ on } \mathcal{A}_g(\mathbb{C})\,.$$

Our moduli space can be compactified by a toroidal compactification over which the Hodge bundle extends. Then the bundles U_ρ extend too.

By the Koecher principle these sections extend over a toroidal compactification and we find that the space of cusp forms $S_\rho(\Gamma_g)$ consists of the sections that vanish at infinity, that is, on the divisor that is added to $\mathcal{A}_g(\mathbb{C})$ to compactify it.

Like for $g = 1$ one can define Hecke operators. They are induced by correspondences and give a commutative algebra of operators. In particular for every prime p we have an operator $T(p)$, but we have also operators $T_i(p^2)$ for $i = 1, \ldots, g$. We refer to [28, 1]. Also the Petersson product has its analogue and we thus can find bases of eigenforms for the action of the Hecke algebra on the spaces of cusp forms.

4 Cohomological interpretation

As alluded to in Section 2 modular forms admit a cohomological interpretation and this is a powerful tool as we shall see. Let $\pi : \mathcal{X}_1 \to \mathcal{A}_1$ be the universal elliptic curve. Then we find a local system $\mathbb{V} = R^1\pi_*\mathbb{Q}$ on $\mathcal{A}_1(\mathbb{C})$ of $\mathbb{Q}$-vector spaces with fibre $H^1(E, \mathbb{Q})$ over $[E]$. This is a local system of rank 2 associated to the standard representation of Γ_1. By taking $\mathrm{Sym}^a(\mathbb{V})$, the ath symmetric power of $\mathbb{V}$, we obtain a local system $\mathbb{V}_a$ of rank $a + 1$.

This local system enters in the cohomological interpretation of elliptic modular cusp forms that Eichler and Shimura gave in the 1950s.

Theorem 4.1 (Eichler-Shimura)**.** *For even $a \geqslant 2$ we have*

$$H_c^1(\mathcal{A}_1 \otimes \mathbb{C}, \mathbb{V}_a \otimes \mathbb{C}) \cong S_{a+2}(\Gamma_1) \oplus \overline{S}_{a+2}(\Gamma_1) \oplus \mathbb{C}$$

where H_c^1 stands for compactly supported cohomology and the right hand side displays the mixed Hodge structure of the left hand side. The space $S_{a+2}(\Gamma_1)$ has Hodge weight $(a+1,0)$ and $\overline{S}_{a+2}$ is the complex conjugate of S_{a+2}.

The case $a = 0$ corresponds to the compactly supported cohomology of the moduli space $\mathcal{A}_1(\mathbb{C})$. In this case $S_2 = (0)$; but if we replace $\mathrm{SL}(2,\mathbb{Z})$ by a congruence subgroup $\Gamma_0(N)$ then cusp forms of weight 2 might show up and these would give differential forms via $f(\tau) \mapsto f(\tau)d\tau$ and these would contribute to the cohomology of the quotient space $\Gamma_0(N)\backslash\mathfrak{H}$ and its compactification $\overline{\Gamma_0(N)\backslash\mathfrak{H}}$. This is how Eichler and Shimura were led to their result. Then the Galois representation associated to f can be found on the Tate module of the Jacobian of the compactification $\overline{\Gamma_0(N)\backslash\mathfrak{H}}$.

But the moduli space $\mathcal{A}_1$ is defined over $\mathbb{Z}$ and the Eichler-Shimura isomorphism has an ℓ-adic counterpart, due to Deligne [21]. It is embodied in the identity

$$\mathrm{Tr}(T(p), S_{a+2}) = \mathrm{Tr}(F_p, H_c^1(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_a^{(\ell)})) - 1\,, \tag{1}$$

where $\mathbb{V}^{(\ell)}$ denotes the ℓ-adic sheaf $R^1\pi_*\mathbb{Q}_\ell$ and $\mathbb{V}_a^{(\ell)}$ its ath symmetric power, and F_p represents a geometric Frobenius element in the Galois group. A convenient way to see where this identity comes from uses the Euler characteristic

$$e_c(\mathcal{A}_1, \mathbb{V}) = \sum (-1)^i [H_c^i(\mathcal{A}, \mathbb{V}_a)]\,,$$

where we take the class $[H^i]$ of the cohomology group in an appropriate Grothendieck group of mixed Hodge modules or of Galois representations depending on whether we take complex cohomology or ℓ-adic étale cohomology; then the statement is that

$$e_c(\mathcal{A}_1, \mathbb{V}_a) = -S[a+2] - 1\,, \tag{2}$$

where $S[a+2]$ is the Chow motive defined by Scholl associated to the space of cusp forms of weight $a+2$ on $\mathrm{SL}(2,\mathbb{Z})$. The rank of this motive is $2s_{a+2}$ with $s_a = \dim S_{a+2}(\Gamma_1)$. It is cut out by projectors on the cohomology of a symmetric power of the universal elliptic curve over the moduli space and its compactification. Then the ℓ-adic Galois representation associated to $S_{a+2}(\mathrm{SL}(2,\mathbb{Z}))$ is to be found in $H_c^1(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_a^{(\ell)})$; here we use the comparison isomorphisms $H_c^i(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_a^{(\ell)}) \cong H_c^i(\mathcal{A} \otimes \overline{\mathbb{Q}}_p, \mathbb{V}_a^{(\ell)}) \cong H_c^i(\mathcal{A} \otimes \overline{\mathbb{Q}}, \mathbb{V}_a^{(\ell)})$ for $\ell \neq p$ and the surjection of Galois groups $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

The Hecke operators $T(n)$ are defined by correspondences on $\mathcal{A}_1$ and act on the cohomology. Moreover, in characteristic p there is the congruence relation that relates the Hecke correspondence and the correspondence defined by Frobenius and its transpose. The Hecke correspondence $T(p)$ is given by all pairs (E, E') of elliptic curves linked by a cyclic isogeny $E \to E'$ of degree p. But in characteristic p if j and j' are the j-invariants of such E and E' we have $j^p = j'$ or $j'^p = j$.

Then the identity (2) and the congruence relation imply that we can calculate the trace of the Hecke operator $T(n)$ on S_{a+2} from the trace of Frobenius on the cohomology $H^1_c(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_a^{(\ell)})$ via Deligne's identity (1).

From a calculational perspective this might seem pointless since we know generators for the ring of modular forms and we can calculate the action of the Hecke operators explicitly. Moreover, we have an explicit formula for the trace of the Hecke operator $T(p)$, the Eichler-Selberg trace formula, see for example Zagier's appendix in [44]. Nevertheless, on the contrary, our approach is a practical one. Indeed, by the philosophy of Weil we can calculate cohomology by counting points over finite fields. And it turns out that we can use this to obtain information about modular forms. In practice we can do this by the following strategy:

1. make a list of all elliptic curves over $\mathbb{F}_p$ up to isomorphism over $\mathbb{F}_p$;
2. for each elliptic curve E in this list determine $\#E(\mathbb{F}_p)$ and $\#\mathrm{Aut}_{\mathbb{F}_p}(E)$.

We know by Hasse that the number of rational points of an elliptic curve over $\mathbb{F}_p$ satisfies $\#E(\mathbb{F}_p) = p + 1 - \alpha - \overline{\alpha}$ with α an algebraic integer with $|\alpha| = \sqrt{p}$ and $\overline{\alpha}$ its complex conjugate. Then we can calculate the trace of $T(p)$ on S_{a+2} by

$$1 + \mathrm{Tr}(T(p), S_{a+2}) = -\sum_E \frac{\alpha^a + \alpha^{a-1}\overline{\alpha} + \cdots + \overline{\alpha}^a}{\#\mathrm{Aut}_{\mathbb{F}_p}(E)}, \tag{3}$$

where the sum is over the elliptic curves in our list. This formula is the concrete embodiment of Deligne's formula (1). For example for the Fourier coefficients of Δ we have

$$\tau(p) = -1 - \sum_E \frac{\alpha^{10} + \alpha^9\overline{\alpha} + \cdots + \overline{\alpha}^{10}}{\#\mathrm{Aut}_{\mathbb{F}_p}(E)}.$$

Given a prime p we can take the E with the same trace t together and then have to list how often a certain trace $t = \alpha + \overline{\alpha}$ occurs in our list, where we count the frequency as

$$w(t) = \sum_{E\colon \#E(\mathbb{F}_p)=p+1-t} \frac{1}{\#\mathrm{Aut}_{\mathbb{F}_p}(E)},$$

where each E with given trace counts with weight $1/\#\mathrm{Aut}_{\mathbb{F}_p}(E)$. For example for $p = 17$ we thus get the list

t	± 8	± 7	± 6	± 5	± 4	± 3	± 2	± 1	0
w	1/4	1/2	3/2	1/2	1	3/2	7/4	1/2	2

With this list we can calculate the trace of $T(17)$ on the space S_k of cusp forms *for all weights* $k \geqslant 4$ by formula (3). Moreover, by having the list for a certain prime, computing the trace for an arbitrary weight is (almost) immediate. For prime powers q we have a slightly modified formula.

As a final remark we note that the 1 in formula (3) comes from the Eisenstein series E_{a+2}. Indeed, for even $a \geqslant 2$ we have

$$H^1_!(\mathcal{A}_1, \mathbb{V}_a) = S[a+2]$$

where $H^i_!$ is the image of H^i_c in H^i. The -1 appearing in (1) and (2) comes from part of the eigenvalue $1 + p^{a+1}$ of E_{a+2} and it appears in the kernel of the map $H^1_c \to H^1$; the other part p^{a+1} appears in the cokernel.

5 Degree two

Since our knowledge about Siegel modular forms of degree $g \geqslant 2$ is much more limited there is every reason to try to generalize this approach to higher g. We consider then

$$\pi : \mathcal{X}_g \to \mathcal{A}_g$$

the universal abelian variety over our moduli space. This yields a local system $\mathbb{V} = R^1\pi_*\mathbb{Q}$ of rank $2g$ on $\mathcal{A}_g(\mathbb{C})$ and its ℓ-adic variant $\mathbb{V}^{(\ell)}$ on $\mathcal{A}_g \otimes \mathbb{C}$ and on $\mathcal{A}_g \otimes \mathbb{F}_p$. The fibre over the point $[X]$ of a principally polarized abelian variety X is $H^1(X, \mathbb{Q})$ or $H^1_{\mathrm{et}}(X, \mathbb{Q}_\ell)$. We shall simply write $\mathbb{V}$ for $\mathbb{V}^{(\ell)}$. From this local system we can construct other local systems as follows. For every irreducible representation of $\mathrm{Sp}(2g, \mathbb{Q})$ of highest weight $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_g)$ we get an associated local system $\mathbb{V}_\lambda$; so with our conventions we have

$$\mathbb{V}_{1,0,\ldots,0} = \mathbb{V}\,.$$

We consider then the cohomology of this local system. As before, a convenient way to deal with it is by using the so-called motivic Euler characteristic

$$e_c(\mathcal{A}_g(\mathbb{C}), \mathbb{V}_\lambda \otimes \mathbb{C}) = \sum_i (-1)^i [H^i_c(\mathcal{A}_g(\mathbb{C}), \mathbb{V}_\lambda \otimes \mathbb{C})]\,,$$

where H^i_c is the compactly supported cohomology and the brackets refer to the class in a Grothendieck group of mixed Hodge structures. But we can also consider ℓ-adic étale cohomology and in this case we consider H^i_c as a Galois representation and then e_c lives in a Grothendieck group of Galois representations.

It is a result of Faltings [26, 25], but see also [46], that H^i and H^i_c have a mixed Hodge filtration; if we define the *interior* cohomology by

$$H^i_! = \text{image of } H^i_c \text{ in } H^i\,,$$

then $H^i_!$ has a pure Hodge structure. Moreover, it follows from [25] that if λ is *regular* (that is, $\lambda_1 > \lambda_2 > \cdots > \lambda_g$) then if $H^i_! \neq 0$ we must have $i = g(g+1)/2$, the dimension of $\mathcal{A}_g$.

Let us specialize to $g = 2$. Then λ is given as a pair $\lambda = (a, b)$ of integers with $a \geqslant b \geqslant 0$ and according to Faltings we have a Hodge filtration

$$F^{a+b+3} \subset F^{a+2} \subset F^{b+1} \subset F^0 = H^3_!(\mathcal{A}_2(\mathbb{C}), \mathbb{V}_{a,b} \otimes \mathbb{C})\,.$$

A main point is now that we have an interpretation of the first step:

$$F^{a+b+3} \cong S_{a-b,b+3},$$

with $S_{a-b,b+3}$ the space of cusp forms on Γ_2 of weight $(a-b, b+3)$; that is, these modular forms are the sections on $\tilde{\mathcal{A}}_2$ of

$$\mathrm{Sym}^{a-b}(\mathbb{E}) \otimes \det(\mathbb{E})^{b+3} \otimes \mathcal{O}(-D)\,,$$

where D is the divisor added to $\mathcal{A}_2$ to obtain an appropriate toroidal compactification $\tilde{\mathcal{A}}_2$ of $\mathcal{A}_2$.

Based on ample numerical evidence Carel Faber and I formulated a conjecture [24] in 2004 that has now been confirmed completely. For regular local systems ($a > b > 0$) this was done by Weissauer [66] in 2009 and the irregular cases by Petersen [49] in 2013.

Before we state it we remark that it follows from the action of -1_X on an abelian surface X that if $a \not\equiv b \,(\mathrm{mod}\, 2)$ then $e_c = 0$. We thus may restrict ourselves to the case $a \equiv b \,(\mathrm{mod}\, 2)$.

Theorem 5.1. *We have*

$$\mathrm{Tr}(T(p), S_{a-b,b+3}) = -\mathrm{Tr}(F_p, e_c(\mathcal{A} \otimes \overline{\mathbb{F}}_p, \mathbb{V}_{a,b})) + \mathrm{Tr}(F_p, e_{2,\mathrm{extra}}(a,b))$$

with $e_{2,\mathrm{extra}}(a,b)$ a correction term given by

$$s_{a-b+2} - s_{a+b+4}(S[a-b+2]+1)L^{b+1} + \begin{cases} S[b+2]+1 & a \quad \text{even}, \\ -S[a+3] & a \quad \text{odd}. \end{cases}$$

Here $s_n = \dim S_n(\Gamma_1)$ and L stands for the Lefschetz motive ($= h^2(\mathbb{P}^1)$). And in order to make it work for $a = 0$ we should define

$$S[2] = -L - 1 \quad \text{and} \quad s_2 = -1\,.$$

We arrived at this conjecture by applying the strategy mentioned for the case $g = 1$: first note that the moduli space of stable curves of genus 2 of compact type coincides with the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. We then make a list of all stable curves of compact type of genus 2 over our finite field $\mathbb{F}_q$ up to isomorphism over that finite field and for each Jacobian X of such a curve we compute the order of the group of automorphisms defined over $\mathbb{F}_q$ and the eigenvalues $\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2$ of Frobenius acting on its ℓ-adic Tate module (or on $H^1(X \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)$) for a prime ℓ different from the characteristic. Then the trace of Frobenius on the cohomology of the local system $\mathbb{V}_{a,b}^{(\ell)}$ is given by summing over the X in our list the values of a Schur function in the eigenvalues $\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2$, that is, the $g = 2$ analogue of (2). For each q this will give a number and we then tried to interpret this number as a function of q as a polynomial plus eigenvalue of an elliptic modular form for small values of a and b. This gave us an idea about the extra term $e_{2,\mathrm{extra}}(a,b)$.

In order to make the list and in order to avoid the calculational problem of determining the order of the automorphism group what we do is the following. We take a suitable family of curves lying finitely over the moduli space and then we count in this family and divide the numbers by the degree of the map to the moduli space.

The way we arrived at the conjecture is a good example of experimental mathematics. The fact that we had a dimensional formula for the spaces of Siegel modular forms of degree 2, due to Tsushima [63], was very helpful. We also knew the numerical Euler characterics of the local systems.

If for given q we have made such a list then this result enables us to compute $\mathrm{Tr}(T(q), S_{a-b,b+3})$ for *all values of* (a, b) ! We did this for all $q < 37$ (and have extended this now to $q < 200$).

Even for classical Siegel modular forms this gave information without reach before. For example, the space $S_{0,35}$ of scalar-valued Siegel modular forms of degree 2 and weight 35 is 1-dimensional. It is generated by a cusp form χ_{35} constructed by Igusa. It occurs in the cohomology of the local system $\mathbb{V}_{32,32}$ where $e_{2,\mathrm{extra}}$ has the form

$$e_{2,\mathrm{extra}}(32, 32) = 5\, L^{34} + S[34]$$

and one finds as eigenvalue $\lambda(p)$ of $T(p)$ for $p \leqslant 37$:

p	$\lambda(p)$ on $S_{0,35}$
2	−25073418240
3	−11824551571578840
5	9470081642319930937500
7	−10370198954152041951342796400
11	−8015071689632034858364818146947656
13	−20232136256107650938383898249808243380
17	118646313906984767985086867381297558266980
19	2995917272706383250746754589685425572441160
23	−1911372622140780013372223127008015060349898320
29	−2129327273873011547769345916418120573221438085460
31	−157348598498218445521620827876569519644874180822976
37	−47788585641545948035267859493926208327050656971703460

Or consider the form $E_8\chi_{35}$ that generates the 1-dimensional space $S_{0,43}$. We have

$$e_{2,\mathrm{extra}}(40, 40) = S[42] + 7\, L^{42}\,.$$

Here we find the following eigenvalues $\lambda(p)$ of $T(p)$:

p	$\lambda(p)$ on $S_{0,43}$
2	−4069732515840
3	−65782425978552959640
5	−44890110453445302863489062500
7	−19869584791339339681013202023932400
11	4257219659352273691494938669974303429235064
13	1189605571437888391664528208235356059600166220
17	−1392996132438667398495024262137449361275278473925020
19	−155890765104968381621459579332178224814423111191589240
23	−128837520803382146891405898440571781609554910722934311120
29	4716850092556381736632805755807948058560176106387507397101740
31	3518591320768311083473550005851115474157237215091087497259584
37	−80912457441638062043356244171113052936003605371913289553380964260

For vector-valued modular forms it is just as powerful. We illustrate this by giving the Hecke eigenvalues for two vector-valued modular cusp forms, namely of weight $(14,7)$ and $(4,17)$. In both cases the dimension of the space of Siegel modular cusp forms is 1. We have

$$e_{2,\mathrm{extra}}(18,4) = -L^5(S[16]+1)+2,$$

and

$$e_{2,\mathrm{extra}}(18,14) = -3L^{15}+S[16]+1\,.$$

The graphs formed by the digits illustrate the growth of the eigenvalues as roughly $p^{(j+2k-3)/2}$. They also can be used to illustrate the fact that these eigenvalues tend to be "smooth" (highly divisible) numbers for small values of (j,k).

p	$\lambda(p)$ on $S_{14,7}$	$\lambda(p)$ on $S_{4,17}$
2	−3696	−266112
3	511272	−210323304
5	118996620	668111687100
7	−82574511536	−420920757352592
11	5064306707064	−388201474991129976
13	−29379924792548	28107151225966031596
17	170082580670244	−3760611385645410867612
19	3752454431256520	−6080023439267575397000
23	−79555863361862928	168303583255503998515536
29	−81010055585118660	15109310600861660971695180
31	−515521596253351616	33344471645582702957462464
37	−40280723363343088436	1247592679027009407366180988

Here we go only till $p \leqslant 37$, but in fact, we have extended the calculations to all $q < 200$. The data will be made available on a website.

As an application of the availability of all these data we can mention the paper [6] where all these data (traces for $q < 150$) are used to approximate critical L-values used in checking congruences.

In the case of degree 2 the analogue of the result of Deligne on the existence of a semi-simple ℓ-adic Galois representation for a Hecke eigenform can be deduced from work of Taylor, Weissauer and Laumon [62, 65, 45]. The dimension of the representation is 4. But unlike the situation for $g = 1$ here it is not necessarily true that the four eigenvalues of a Frobenius element at p have a fixed absolute value p^w for some w. When this was discovered this fact came as a surprise.

In fact, at the end of the 1970s Kurokawa and Saito discovered (see [42]) that there are Siegel modular forms of degree 2 that are lifts from elliptic modular forms. In fact, if $f = \sum_n a(n)q^n$ is an elliptic cusp form of weight $2k-2$ with k even and a normalized (i.e. $a(1) = 1$) eigenform for the Hecke algebra, then there is a scalar-valued Siegel modular form of even weight k on Γ_2, also an eigenform for the Hecke algebra, with Hecke eigenvalues

$$\lambda(p) = p^{k-2} + a(p) + p^{k-1} \,. \tag{4}$$

Note that $a(p) = \alpha + \bar{\alpha}$ with $|\alpha| = p^{k-3/2}$. So there is a 4-dimensional Galois representation corresponding to (4) which we can realize as

$$\mathbb{Q}(-k+2) \oplus R_f \oplus \mathbb{Q}(-k+1)\,,$$

where R_f is the 2-dimension representation $\rho_{\ell,f}$ associated to f by Deligne. But for reasons of cohomological weight only the 2-dimensional part R_f defined by f can occur in the interior (middle-dimensional) cohomology of the local system $\mathbb{V}_{k-3,k-3}$ on $\mathcal{A}_2$. So the full 4-dimensional Galois representation is not to be found in the cohomology of the local system, only a part.

These lifts found by Kurokawa and Saito lie in a subspace of $S_k(\Gamma_2)$ called the Maass subspace consisting of Siegel modular forms whose Fourier series

$$\sum_{N \geqslant 0} a(N) e^{2\pi i \mathrm{Tr}(N\tau)}$$

has the property that for $N = (n, r/2; r/2, m)$ the coefficient depends only on the discriminant $d(N) = 4mn - r^2$ and the content g.c.d(n, r, m). The $1-1$ correspondence between the Hecke eigenforms in $S_{2k-2}(\Gamma_1)$ and Hecke eigenforms in $S_k(\Gamma_2)$ was proved to a large extent by Maass, and completed by Andrianov and Zagier, see [67].

In joint work with Bergström and Faber [7] we extended all this to the case of degree 2 and level 2. Here the group Γ_2 is replaced by the kernel $\Gamma_2[2] = \ker(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}))$ of the natural reduction map. Since $\mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}) \cong \mathfrak{S}_6$, the symmetric group on six letters, everything comes with an action of this group. In [7] we formulated the conjectural analogue of the above theorem. The cohomology and spaces of modular forms appear as representation spaces of $\mathfrak{S}_6$. Although it is conjectural it is based on very firm evidence. In [52] Mirko Rösner announced a proof of our conjectures.

In joint work with Fabien Cléry and Sam Grushevsky [18] we used these conjectural results to determine the structure of a number of $M(\Gamma_2[2])$-modules of vector-valued modular forms. By work of Igusa we know that the ring $M^{\rm ev}(\Gamma_2[2])$ of even-weight scalar-valued modular forms on $\Gamma_2[2]$ is generated by a 5-dimensional vector space of modular forms of weight 2 that form a representation of type $[2,1,1,1,1]$ for the group $\mathfrak{S}_6$ satisfying a quartic relation. The ring of modular forms of all weights is a quadratic extension generated by a modular form of weight 5. Then the direct sums

$$\bigoplus_k M_{j,k}(\Gamma_2[2])$$

for fixed value of j are modules over $M(\Gamma_2[2])$. We determined the structure of such modules for small values of j. The conjectural results suggested where to find generators of these modules and by working with explicit generators and using the dimension formulas the structure could be determined. We showed that the $M(\Gamma_2[2])$-module

$$\Sigma_2 = \bigoplus_{k:\text{odd}} S_{2,k}(\Gamma_2[2])$$

is generated by ten modular forms $\Phi_i \in S_{2,5}$ $(i = 1, \ldots, 10)$ with $\sum \Phi_i = 0$ that span a 9-dimensional $\mathfrak{S}_6$-representation of type $[2,2,1,1]$. Since we know the decomposition of $S_{2,k}$ as a $\mathfrak{S}_6$-representation we can read off the relations. Similarly, the module

$$\bigoplus_{k:\text{even}} M_{2,k}(\Gamma_2[2])$$

is generated by 15 modular forms $G_{ij} \in S_{2,4}(\Gamma_2[2])$ with $1 \leqslant i < j \leqslant 6$ that generate a $\mathfrak{S}_6$-representation of type $[3,1,1,1] \oplus [2,1,1,1,1]$.

Before we move to degree 3 we remark that we can formulate the result for genus 2 differently, namely in the form

$$e_c(\mathcal{A}_2, \mathbb{V}_\lambda) = -S[a-b, b+3] + e_{2,\text{extra}}(a,b) \tag{5}$$

in analogy with the case $g = 1$, by assuming that there is a motive $S[a-b, b+3]$ of rank $4 \dim S_{a-b,b+3}$ with the property that

$$\begin{aligned}\operatorname{Tr}(T(p), S_{a-b,b+3}) &= \operatorname{Tr}(F_p, S[a-b,b+3]) \\ &= -\operatorname{Tr}(F_p, e_c(\mathcal{A}_2 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_{a,b}) + e_{2,\text{extra}}(a,b)).\end{aligned}$$

The motive $S[a-b, b+3]$ is still hypothetical, but would be the analogue of the motive $S[a+2]$ constructed by Scholl. But even if we do not know whether such a motive exists we can use $S[a-b, b+3]$ in the formulation above just as a bookkeeping device with the property that

$$\operatorname{Tr}(T(p), S_{a-b,b+3}) = \operatorname{Tr}(F_p, S[a-b, b+3])\,.$$

6 Degree three

Now we move to degree $g = 3$. In this case the local systems are indexed by $\lambda = (a, b, c)$ with integers $a \geqslant b \geqslant c \geqslant 0$. The goal is now to find an analogue of the formula (5); in other words the problem is to come up with a formula for $e_{3,\mathrm{extra}}(a, b, c)$. In [8] we formulated a conjecture.

Conjecture 6.1. *We have*

$$e_c(\mathcal{A}_3 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_\lambda) = S[a-b, b-c, c+4] + e_{3,\mathrm{extra}}(a, b, c)$$

with the correction term $e_{3,\mathrm{extra}}(a, b, c)$ *defined by*

$$\begin{aligned} e_{3,\mathrm{extra}}(a, b, c) = &- e_c(\mathcal{A}_2 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_{a+1,b+1}) - e_{2,\mathrm{extra}}(a+1, b+1) \otimes S[c+2] \\ &+ e_c(\mathcal{A}_2 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_{a+1,c}) + e_{2,\mathrm{extra}}(a+1, c) \otimes S[b+3] \\ &- e_c(\mathcal{A}_2 \otimes \overline{\mathbb{F}}_p, \mathbb{V}_{b,c}) - e_{2,\mathrm{extra}}(b, c) \otimes S[a+4] \end{aligned}$$

and where $S[a-b, b-c, c+4]$ *denotes a hypothetical motive or bookkeeping device such that*

$$\mathrm{Tr}(T(p), S_{a-b,b-c,c+4}) = \mathrm{Tr}(F_p, S[a-b, b-c, c+4]) \,.$$

Again we arrived at this conjecture by using the fact that in genus 3 the moduli of principally polarized abelian varieties are close to the moduli of genus three curves. In fact, the map $\mathcal{M}_3 \to \mathcal{A}_3$ is of degree 2 as a map of stacks. Then we can calculate the trace of a power of Frobenius on the cohomology of $\mathbb{V}_\lambda$ by making a list of all principally polarized abelian varieties of dimension 3 over $\mathbb{F}_q$ up to isomorphism over $\mathbb{F}_q$ that are Jacobians of curves and calculating the order of the automorphism group and the eigenvalues of Frobenius. By summing a Schur function in these eigenvalues over this list (weighted by the inverse orders of the automorphism groups) we find the trace. Again we then tried to interpret this number as a function of q in terms of "known" functions, like polynomials in q, traces of Hecke operators on spaces of cusp forms of degree 1 and 2. That is how we arrived at the expressions for $e_{3,\mathrm{extra}}(a, b, c)$. In fact, the form of part of the Eisenstein cohomology found in [30] guided our guesses and suggested a first approximation. Surprisingly the formula for degree 3 looks simpler than the formula for degree 2. We did not know the dimension of the spaces of cusp forms, but we knew the numerical Euler characteristics E_c (obtained by replacing the terms in the conjecture by their dimensions) of the local systems by [13].

The evidence for this conjecture is considerable:

1. While we are summing rational numbers, since we divide by the order of the automorphism groups, the procedure produces integral values for the traces of the Hecke operators.

2. What we know about the dimensions of the spaces of modular forms agrees with the conjecture; for example, if we know that the space of cusp forms vanishes we always find that $\mathrm{Tr}(e_c) = \mathrm{Tr}(e_{\mathrm{extra}})$.

3. We know the numerical Euler characteristics E_c by [13]; as it turns out $E_c - E_{\text{extra}}$ is always divisible by 8.

4. In all vector-valued cases when $E_c = E_{\text{extra}}$ (i.e. when we conjecture that the space of cusp forms vanishes), we have $\text{Tr}(e_c) = \text{Tr}(e_{\text{extra}})$.

5. Recently Taïbi [61] has been able to calculate dimensions of Siegel modular forms assuming the validity of Arthur's trace formula. His values fit ours.

6. There is consistency with the results of Chenevier and Renard [15] on the number of automorphic level one polarized algebraic regular automorphic representation of GL_n over $\mathbb{Q}$ for $n \leqslant 8$.

7. Harder type congruences between elliptic modular forms and Siegel modular forms, see below.

8. Mégarbané has computed a number of eigenvalues (up to $q = 13$) for several Siegel modular forms of degree 3 as they appear among the automorphic forms of SO(8). A number of consistency checks have been made.

9. There is agreement with the eigenvalues computed in [17].

Let us give two examples. We have

$$e_c(\mathcal{A}_3, \mathbb{V}_{11,5,2}) = S[6,3,6] - S[12]L^3 + L^7 - L^3 + 1\,,$$

where $S[6,3,6]$ stands as above for a bookkeeping device. In this case we should have $\dim S_{6,3,6} = 1$. The other example is for weight $(4,2,8)$. Again we should have $\dim S_{4,2,8} = 1$. We have

$$e_c(10,6,4) = L^8 + S[4,10] + S[4,2,8] - 1\,.$$

The corresponding modular form is of type G_2. We have the following table of eigenvalues (where for powers of a prime we use the convention of [8]):

p	$\lambda(q)$ on $S_{6,3,6}$	$\lambda(q)$ on $S_{4,2,8}$
2	0	9504
3	−453600	970272
4	10649600	89719808
5	−119410200	−106051896
7	12572892800	112911962240
8	0	1156260593664
9	−29108532600	5756589166536
11	−57063064032	44411629220640
13	−25198577349400	209295820896008
16	341411782197248	−369164249202688
17	−107529004510200	1230942201878664
19	1091588958605600	51084504993278240

Our results for $g = 3$ also tell what kind of lifts one finds. We even could identify lifts that come from the Lie group G_2 as suggested by Gross and Savin. We refer to [8, 9.1].

7 Harder type congruences

Congruences between modular forms have a considerable history. There are well-known congruences between cusp forms and Eisenstein series like the congruence between the Hecke eigenvalues

$$\tau(p) \equiv p^{11} + 1 \pmod{691}$$

for Δ and the Eisenstein series of weight 12. Such congruences occur for primes dividing the numerator of $B_k/2k$ with B_k the kth Bernoulli number, see [22]. Swinnerton-Dyer determined all congruences modulo a prime for modular forms on $\mathrm{SL}(2,\mathbb{Z})$, see [60, 56].

Kurokawa found several congruences between Siegel modular forms in 1979 in [43]. In particular, he conjectured the following. Suppose that a prime ℓ divides the critical value $2k-2$ of the symmetric square L-series of an eigenform f of weight k on $\mathrm{SL}(2,\mathbb{Z})$. Then there should exist a cusp form F of degree 2, an eigenform of the Hecke operators, whose eigenvalues for $T(p)$ are congruent to those of the weight k Klingen Eisenstein series modulo ℓ for every p. Some of these congruences were proven by Mizumoto in 1986, see [48] and reconsidered by Katsurada and Mizumoto [41, 23].

In general if there is a Hecke invariant splitting of the vector space of Siegel modular forms, one might expect congruences between the factors of this splitting. Harder suspected congruences between cusp forms and Saito-Kurokawa lifts. He was motivated by his work on the Eisenstein cohomology of Γ_2.

The moduli space $\mathcal{A}_g$ admits a "minimal" compactification $\mathcal{A}_g^*$ obtained by mapping the quotient $\Gamma_g\backslash\mathfrak{H}_g$ into projective space by using a basis of the space $M_k(\Gamma_g)$ of scalar-valued modular forms of sufficiently high weight k and then taking the closure. In other words, by taking Proj of the graded ring of scalar-valued Siegel modular forms. Set-theoretically we have

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0 \, .$$

It is usually called the Satake or Baily-Borel compactification. This decomposition is mirrored in the Siegel operator that associates to a Siegel modular form of degree g a form of degree $g-1$ by the limiting procedure

$$\Phi_g f(\tau') = \lim_{t\to\infty} f \begin{pmatrix} \tau' & 0 \\ 0 & it \end{pmatrix} \qquad \text{with } \tau' \in \mathfrak{H}_{g-1}, t \in \mathbb{R}_{>0} \, .$$

If $f \in M_k(\Gamma_g)$ then $\Phi_g(f) \in M_k(\Gamma_{g-1})$; if f is a vector-valued Siegel modular form of weight ρ then $\Phi_g f$ is a form of weight ρ' with ρ' the irreducible representation of highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_{g-1})$ if ρ has highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_g)$. The cusp forms are by definition the forms with $\Phi_g f = 0$. So the boundary of our moduli space is the place where modular forms of different degrees interact. Using the interpretation with cohomology one sees the interaction in the long exact sequences connecting cohomology on the interior and on the boundary. Harder studied this in an extensive way using the Borel-Serre compactification (a real manifold with corners) and Betti cohomology. He considers the cohomology in the local system of $\mathbb{Z}$-modules $\mathbb{V}_\lambda^{\mathbb{Z}}$ and over a ring $R = \mathbb{Z}[1/N]$ with an appropriate integer N inverted, one considers the long exact cohomology sequence

$$H_c^3(\Gamma_2\backslash\mathfrak{H}_2, \mathbb{V}_\lambda^{\mathbb{Z}} \otimes R) \to H^3(\Gamma_2\backslash\mathfrak{H}_2, \mathbb{V}_\lambda^{\mathbb{Z}} \otimes R) \to H^3(\partial(\overline{\Gamma_2\backslash\mathfrak{H}_2}), \mathbb{V}_\lambda^{\mathbb{Z}} \otimes R) \,.$$

Using the eigenvalues of the Hecke operators that have different absolute values on $H_!^3$ and on the Eisenstein part, here taken with $\mathbb{Q}$-coefficients, one sees that there is a direct sum decomposition of $H^3(\Gamma_2\backslash\mathfrak{H}_2, \mathbb{V}_\lambda)$ as $H_!^3(\Gamma_2\backslash\mathfrak{H}_2, \mathbb{V}_\lambda)$ plus its image in $H^3(\partial(\overline{\Gamma_2\backslash\mathfrak{H}_2}), \mathbb{V}_\lambda)$. This latter part contributes to the Eisenstein cohomology and the classes there can be described by certain Eisenstein series. For an Eisenstein series generating an eigen space of the cohomology the constant term is a product of critical values of L-functions associated to a modular form on $\mathrm{SL}(2,\mathbb{Z})$. But if one works with integral cohomology (Harder uses Betti cohomology) and intersects this with the direct sum of the interior integral cohomology and the integral Eisenstein cohomology it might be that this direct sum has a non trivial index in the integral cohomology $H^3(\Gamma_2\backslash\mathfrak{H}_2, \mathbb{V}_\lambda^{\mathbb{Z}})$. This index is related to primes dividing this critical values of the L-functions occuring in the denominator of the constant term of this Eisenstein series. This will lead to a congruence between a part of the interior cohomology and part of the Eisenstein cohomology. In concrete terms it leads to a congruence between the eigenvalues of an honest Siegel modular form of genus 2 (that is, not a Saito-Kurokawa lift) and the eigenvalues of an elliptic modular form.

Harder had this idea already quite early, see his Lecture notes [36], but it seemed difficult to check it. We quote “Ich halte es für sehr interessant, numerische Rechnungen durchzuführen. ... Ich glaube, daß die Implementierung hiervon auf einem Computer, einige schwierige Aufgabe darstellt. Sie wird viel Zeit und Rechenaufwand kosten ...” ([36, p. 101]). But more than ten years later when it became feasible to calculate the eigenvalues of examples of Siegel modular forms, Harder set himself the task of making such conjectures explicit. The first case he arrived at was the conjectured congruence for an eigenform in F in $S_{4,10}(\Gamma_2)$ and the normalized elliptic cusp form $f = \sum a(n)q^n$ generating $S_{22}(\Gamma_1)$: if $\lambda(p)$ denotes the eigenvalue of F for the Hecke operator $T(p)$ we should have

$$\lambda(p) \equiv p^8 + a(p) + p^{13} \pmod{41} \tag{6}$$

for all primes p. The eigenvalues we could calculate fitted perfectly, see Harder's report on this in [37]. Harder formulated a rather precise conjecture and in many examples the congruences he predicted could be confirmed numerically, see [29]. For other numerical checks see [31]. Some of the congruences have now been proved, for example the original congruence (6) modulo 41 for weight $(4,10)$ by Chenevier and Lannes [14] by their work on unimodular lattices of rank 24 and the modular forms related to this.

Harder's conjectures admit extensions to higher genus. The underlying idea is that direct sum decompositions of rational cohomology that are stable under the action of the Hecke algebra do not necessarily hold for integral cohomology and this leads to congruences. These congruences are predicted by divisibility properties of critical values of L-functions associated to modular forms.

Since we have various Hecke invariant subspaces of the cohomology, like the interior cohomology, Eisenstein cohomology, spaces of lifted forms, we can expect in higher degree many different types of congruences.

In our paper [8] we formulated a number of extensions to degree three Siegel modular forms. Let us give just one example. The conjecture says in this case that if we have eigenforms $f \in S_{c+2}(\Gamma_1)$ and $g \in S_{a+b+6}(\Gamma_1)$ and an ordinary prime ℓ (in the field of eigenvalues of f and g) dividing the critical value at $s = a + c + 5$ of the L-series belonging to $\mathrm{Sym}^2(f) \otimes g$, then there should be a genuine Siegel eigenform $F \in S_{a-b,b-c,c+4}(\Gamma_3)$ such that

$$\lambda_F(p) \equiv \lambda_f(p)(p^{b+2} + \lambda_g(p) + p^{a+3}) \bmod \ell$$

for all primes p. An example is the case $(a, b, c) = (13, 11, 10)$ with $F \in S_{2,1,14}(\Gamma_3)$. The Euler characteristic is

$$2(L^{13}S[12] + 1) - 2L^{11} + S[2, 1, 14]$$

and $\dim S_{2,1,14} = 1$. The forms are $f = \Delta = q - 24q^2 + \cdots$ in S_{12} and $g \in S_{30}$ with Fourier series $g = 1 + (4320 + 96\sqrt{51349})\, q + \cdots$ and the prime dividing critical value of the L-series turns out to be $\ell = 199$. The heuristic traces of $T(p)$ for $F \in S_{2,1,14}$ are in the table below. One checks that the congruence above is indeed satisfied for all $q \leqslant 17$, for example for $p = 2$ the norm of

$$-2073600 + 24\,(2^{13} + 4320 + 96\sqrt{51349} + 2^{16})$$

equals -232402452480 and is divisible by $\ell = 199$.

p	$\lambda(q)$ on $S_{2,1,14}$
2	−2073600
3	−1885952160
4	1080940298240
5	−26851408810200
7	−34909007533294720
8	−5890898142638899200
9	2339767572242234760
11	−660044916805998490272
13	−10848446812874015943640
16	461697465916451767451648
17	−2009932545573210270768120
19	17632053727783741943750240

For further discussion of congruences between modular forms we refer to the paper by Bergström and Dummigan [5] where a vast generalization of Harder's conjecture is made. For finding this generalization the data provided by our calculations were of great help.

8 Picard modular forms

The approach that we are advocating in this paper can be applied not only to Siegel modular forms, but to other types of modular forms as well. Shimura gave in 1964 a list of rational ball quotients [58], that is, arithmetic ball quotients birationally equivalent to a projective space. These ball quotients in question turn out to be moduli spaces of curves that are covers of the projective line of given degree and therefore are amenable to our approach. Bergström and I have pursued this for two cases of 2-dimensional ball quotients. These ball quotients are Picard modular surfaces.

If F is an imaginary quadratic field with ring of integers O_F and discriminant D we consider the vector space $V = F^3$ with hermitian form h

$$z_1\bar{z}_2 + z_2\bar{z}_1 + z_3\bar{z}_3$$

where we denote the Galois automorphism of F by $\alpha \mapsto \bar{\alpha}$ and we let

$$G = \{g \in \mathrm{GL}(3, F) : h(gz, gw) = \eta(g)h(z, w)\}$$

with $\eta(g) \in \mathbb{Q}$ the so-called multiplier. Then G is an algebr[illegible]oup defined over $\mathbb{Q}$. If $G^0 = \ker \eta$ we set $\Gamma = G^0(\mathbb{Z})$ and $\Gamma_1 = G^0(\mathbb{Z}) \cap \ker \det$ [illegible]lement $g \in G(\mathbb{R})$ with $\eta(g) > 0$ acts on

$$B = \{l \in V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{R} : l \text{ a line with } h_{|l} < 0\},$$

the set of lines in the 3-dimensional complex vector space $V_{\mathbb{C}}$ on which h is negative definite. It is a complex 2-ball that can be identified with

$$B = \{(u, v) \in \mathbb{C}^2 : v + \bar{v} + u\bar{u} < 0\},$$

where $u = z_3/z_2$ and $v = z_1/z_2$. The space $\Gamma\backslash B$ is an orbifold. It is not compact but can be compactified by adding finitely points, called the cusps. These are singular and for a freely acting subgroup Γ' of finite index of Γ the cusps of $\Gamma'\backslash B$ can be resolved by elliptic curves. The quotients $\Gamma\backslash B$ are moduli spaces of 3-dimensional abelian varieties with endomorphisms from O_F and the action of O_F on the tangent space of an abelian threefold has type $(2, 1)$.

Picard studied these quotients and the corresponding modular forms already at the end of the nineteenth century, see [50, 51]. Shimura took up their study again in the 1960s and in the 1970s Shintani wrote an unpublished paper [59] about the (vector-valued) modular forms associated to these groups. Later Shiga, Holzapfel, Feustel, Finis and others studied these 2-dimensional spaces and the associated scalar-valued modular forms, see [39, 27].

We looked at two cases in Shimura's list of ball quotients; they have $F = \mathbb{Q}(\sqrt{-1})$ or $F = \mathbb{Q}(\sqrt{-3})$.

We restrict ourselves here to the case $F = \mathbb{Q}(\sqrt{-3})$ and put

$$\Gamma[\sqrt{-3}] = \{g \in \Gamma : g \equiv 1 \ (\mathrm{mod} \sqrt{-3})\}$$

and

$$\Gamma_1[\sqrt{-3}] = \{g \in \Gamma[\sqrt{-3}] : \det(g) = 1\} .$$

Then in this case the moduli space $\Gamma_1[\sqrt{-3}]\backslash B$ is a moduli space of curves: Galois covers of genus 3 and degree 3 of $\mathbb{P}^1$ with a marking of the ramification points. These curves can (generically) be written as

$$y^3 = f(x)$$

with f a polynomial of degree 4 with non-vanishing discriminant.

The surface $\Gamma_1[\sqrt{-3}]\backslash B$ can be compactified by adding four cusps. It comes equipped with two vector bundles: we have on B an exact sequence

$$0 \to L \longrightarrow V_{\mathbb{C}} \longrightarrow Q \to 0 ,$$

where L denotes the tautological line bundle and Q the tautological quotient of rank 2 and these bundles descend to $\Gamma_1[\sqrt{-3}]\backslash B$ and extend over the compactification. With $U = Q^\vee$ we have the isomorphisms

$$U \otimes L \cong \Omega^1(\log D), \qquad L^3 \cong \Omega^2(\log D) ,$$

where D is the divisor that resolves the four cusps. Correspondingly, we have two factors of automorphy given for $g = (g_{ij})$ in our group by

$$j_1((u,v),g) = g_{21}u + g_{22}v + g_{23} ,$$

corresponding to L, and

$$j_2((u,v),g) = \mathrm{Jac}((u,v),g)^{-t} \cdot j_1((u,v),g)^{-1}$$

with Jac the Jacobian of the action, corresponding to U. This gives us the notion of a Picard modular form. A scalar-valued modular form of weight k is a holomorphic section of L^k and a scalar-valued cusp form of weight k is a section of $\Omega^2 \otimes L^{k-3}$. The ring of scalar-valued Picard modular forms in this case is known by work of Shiga, Holzapfel and Feustel. We have

$$M(\Gamma[\sqrt{-3}]) = \bigoplus_k M_k(\Gamma[\sqrt{-3}]) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2] ,$$

where the φ_i are modular forms of weight 3 and

$$M(\Gamma_1[\sqrt{-3}]) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2, \zeta]/(\zeta^3 - P(\varphi_0, \varphi_1, \varphi_2))$$

with ζ a cusp form of weight 6 and P a polynomial of degree 6. This form ζ plays a role analogous to that played by Δ for $\mathrm{SL}(2,\mathbb{Z})$. In 1998 Finis (see [27]) calculated some eigenvalues under the Hecke operators of low-weight Picard modular forms for $M(\Gamma_1[\sqrt{-3}])$.

We also have vector-valued modular forms; the cusp forms correspond to sections of

$$\mathrm{Sym}^j(U) \otimes \Omega^2 \otimes L^{k-3}.$$

Although Shintani's unpublished paper treats vector-valued Picard modular forms there were no explicit examples in the literature before we considered this case. Since our quotient parametrizes curves our cohomological approach also works here. We have a universal family

$$\pi : \mathcal{X} \longrightarrow \mathcal{M}$$

of curves of genus 3 with an action of a group of order 3. Let ρ be a fixed nontrivial third root of 1. The local system $\mathbb{V} = R^1\pi_*\mathbb{Q}$ is of rank 6 and it or its étale ℓ-adic variant splits after base change to F as

$$\mathbb{W} \oplus \mathbb{W}',$$

where $\mathbb{W}$ (resp. $\mathbb{W}'$) corresponds to the ρ-eigenspace (resp. ρ^2-eigenspace) under the action of the group of order 3 and $\mathbb{W}$ is of rank 3. Note that after base change we have $G \sim \mathrm{GL}(3, F) \times \mathbb{G}_m$, where the multiplicative group $\mathbb{G}_m$ corresponds to the multiplier η. The local systems that we can consider here correspond to irreducible representations of this group.

We consider local systems $\mathbb{W}_\lambda$ with $\lambda = (a + i, i, -b + i)$ occurring in

$$\mathrm{Sym}^a\mathbb{W} \otimes \mathrm{Sym}^b\mathbb{W}' \otimes \det \mathbb{W}^i,$$

where $\det \mathbb{W}$ is a local system the 6th power of which is a constant local system. So we may choose i modulo 6 (or even modulo 3). We look at the motivic Euler characteristic

$$e_c(\mathcal{M}, \mathbb{W}_\lambda) = \sum_i (-1)^i [H^i_c(\mathcal{M}, \mathbb{W}_\lambda)],$$

again taken in an appropriate Grothendieck group of mixed Hodge structures or Galois representations. As above for the Siegel modular case we denote the interior cohomology, that is, the image of compactly supported cohomology in the usual cohomology, by $H^i_!$. We know by work of Schwermer-Ragunathan that $H^i_! \neq (0)$ implies $i = 2$ for regular λ and we can show that we have a Hodge filtration on $H^2_!$ of the form

$$F^{a+b+2} \subset F^{a+1} \subset F^0 = H^2_!(\mathcal{M}, \mathbb{W}_\lambda).$$

As in the Siegel modular case, a main point is here that the first step F^{a+b+2} can be interpreted as a space of cusp forms

$$F^{a+b+2} \cong S_{b,a+3}(\Gamma[\sqrt{-3}], \det{}^i),$$

namely, cusp forms of weight $(b, a + 3)$ and character $\det^i$. Such modular forms can be described as functions $f : B \to \mathrm{Sym}^j(\mathbb{C}^2)$ satisfying

$$f(\gamma(u, v)) = \det(\gamma)^i\, (j_1(\gamma, u, v))^{a+3}\mathrm{Sym}^b(j_2(\gamma, u, v))f(u, v)$$

for $\gamma \in \Gamma[\sqrt{-3}]$ and a vanishing condition at the cusps.

To gauge the compactly supported cohomology of these local systems we can make counts over finite fields of cardinality $q \equiv 1 (\mathrm{mod}\, 3)$. But in order to be able to extract information about the cusp forms from it we have to find the extra term $e_{\mathrm{extra}}(a,b)$ that gives the "noise" coming from the boundary and "endoscopic forms" just as we did in the Siegel modular case. Here the term $e_{\mathrm{extra}}(a,b)$ is composed of two parts: the Eisenstein cohomology and the endoscopic part.

The Eisenstein cohomology is given as

$$e_{\mathrm{Eis}}(\mathcal{M}, \mathbb{W}_\lambda) = \sum_i (-1)^i [\ker(H_c^i \to H^i(\mathcal{M}, \mathbb{W}_\lambda)] \,.$$

This has been determined by Harder [35, 36]. As a Galois representation it is given as an explicit sum of Hecke characters of F. So we subtract it. But then the endoscopy remains. Indeed, there is a plethora of endoscopic terms. In [11] we determined heuristically all those terms. Subtracting these gives us the trace of the Hecke operator on the spaces of "honest" Picard modular forms. In fact, we have a completely explicit but complicated conjectural formula for $e_c(\mathcal{M}, \mathbb{W}_\lambda)$. We refer to [11] and content ourselves here by giving some simple examples.

We note that the symmetric group $\mathfrak{S}_4$ acts on our modular surface (permuting the four cusps) and we describe the cohomology as a representation of the symmetric group $\mathfrak{S}_4$. We denote an irreducible representation of $\mathfrak{S}_4$ by the corresponding partition, thus writing $s[4], \ldots, s[1,1,1,1]$, the first corresponding to the trivial representation and the last to the alternating one.

It turns out that the Eisenstein cohomology vanishes for $\lambda = [a+i, i-b+i]$ with $a \equiv b \equiv 2 (\mathrm{mod}\ 3)$ and $i = 1, 2$. Take $\lambda = [6k, 0, 0]$. Then the result is

$$e_!(\mathcal{M}, \mathbb{W}_\lambda) = k(s[4] + s[3,1] + s[2,2]) L^{6k+1,1} + S[0, 6k+3, \det^2] \,.$$

Here $L^{\alpha,\beta}$ stands for a 1-dimensional Galois representation of $\mathrm{Gal}(\overline{F}/F)$ or a Hecke character; moreover $S[0, 6k+3, \det^2]$ stands for a hypothetical motive or just as a bookkeeping device with the property that

$$\mathrm{Tr}(F_p, S[0, 6k+3, \det^2]) = \mathrm{Tr}(T(p), S_{0,6k+3}(\Gamma[\sqrt{-3}], \det^2)) \,.$$

Using this and the counting that we did we can calculate traces of Hecke operators $T(p)$ for $p \leqslant 43$ equivariantly for the modular forms that appear here and the results fit for scalar-valued modular forms with the calculations done by Finis. And in fact this has now been extended up to traces of $T(q)$ for $q < 1000$.

We add an example involving vector-valued Picard modular forms: we consider the local system $\mathbb{W}_\lambda$ for λ of the form

$$\lambda = (6k+1, 0, -1)$$

and in this case we find the conjectural formula

$$e_!(\mathcal{M}, \mathbb{W}_\lambda) = (ks[4] + ks[3,1] + (k+1)s[2,2]) L^{6k+2,2} + S[1, 6k+4, \det^2].$$

Assuming this we can calculate the trace of $T(q)$ for $q < 1000$ on the spaces $S_{1,6k+4}(\Gamma[\sqrt{-3}], \det^2)$. In general, for arbitrary λ the formulas are more involved than these due to the many endoscopic terms.

We are still in the process of writing up these results. But in the meantime Fabien Cléry and I have used these heuristic results to investigate modules of vector-valued Picard modular forms over the ring of scalar-valued Picard modular forms [16]. For example, consider

$$\Sigma_1^{(i)} = \bigoplus_{k=1}^{\infty} S_{1,3k+1}(\Gamma[\sqrt{-3}], \det{}^i) \qquad \text{for } i = 0, 1, 2.$$

This is a module over the ring of scalar-valued modular forms

$$M = \bigoplus_k M_{0,3k}(\Gamma[\sqrt{-3}]) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2]\,.$$

Guided by the heuristics obtained by counting curves over finite fields we proved the following result.

Theorem 8.1. *The module Σ_1^0 is generated over the ring M of scalar-valued Picard modular forms by three vector-valued modular forms $\Phi_0, \Phi_1, \Phi_2 \in S_{1,7}(\Gamma[\sqrt{-3}])$ with the relation*

$$\varphi_0\Phi_0 + \varphi_1\Phi_1 + \varphi_2\Phi_2 = 0\,.$$

The modular forms Φ_i appearing here are constructed as a sort of Cohen-Rankin bracket, with for example Φ_2 given by

$$\Phi_2 = [\varphi_0, \varphi_1] \sim \varphi_0 \begin{pmatrix} \partial\varphi_1/\partial u \\ \partial\varphi_1/\partial v \end{pmatrix} - \varphi_1 \begin{pmatrix} \partial\varphi_0/\partial u \\ \partial\varphi_0/\partial v \end{pmatrix}.$$

We can calculate the eigenvalues for these forms and the values fit the heuristic values perfectly. The description of the module makes it possible to construct explicitly eigenforms and calculate eigenvalues and these have been used to confirm the conjectural formulas for $e_c(\mathcal{M}, \mathbb{W}_\lambda)$ of [11]. We just give one table that illustrates the case of the forms Φ_i.

α	p	$\lambda_\alpha(\Phi_i)$
$1+3\rho$	7	$759+261\rho$
$1-3\rho$	13	$-4137+1683\rho$
$-2+3\rho$	19	$24042+14733\rho$
$1+6\rho$	31	$-145401-241830\rho$
$4-3\rho$	37	$12900-114849\rho$
$1-6\rho$	43	$246567-8946\rho$
$4+9\rho$	61	$1048836-173205\rho$
$-2-9\rho$	67	$-1539510-1246887\rho$
$1+9\rho$	73	$-1563729+1261143\rho$
$7-3\rho$	79	$9921297+3294171\rho$
$-8+3\rho$	97	$5678616-3870891\rho$
-2	2	72
-5	5	89622

For the details and the structure of Σ_1^1 and Σ_1^2 and various other modules we refer to [16].

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