Solution Exam Riemann Surfaces (2013)

1) Write $\omega \in \mathcal{E}(Y)$ locally in a coordinate neigborhood (with coordinate z) as $f\,dz$. Then we have

$$d\omega = (\partial f/\partial z)dz \wedge dz + (\partial f/\partial \overline{z})d\overline{z} \wedge dz = (\partial f/d\overline{z})\,d\overline{z} \wedge dz$$

and this is zero if and only if $\partial f/d\overline{z} = 0$, that is, by Cauchy-Riemann, if f is holomorphic.

2) Ramification can occur only at the zeros or poles of f. Let $p \in \mathbb{P}^1$ be a simple zero of f and z a local coordinate at p with $z(p) = 0$. Then f is of the form $z\,u$ with u a non-vanishing function at p. Changing z to zu gives the covering locally as $y^3 = z$ showing that the ramification index is 3. At $z = \infty$ the function f has a pole of order 4; therefore we can write with $w = 1/z$ the covering locally as $y^3 = w^4\,u$ with u a non-vanishing holomorphic function. Changing y to yw we get $y^3 = wu$ and see again that the ramification index is 3. By the Hurwitz-Zeuthen formula we get

$$2g(X) - 2 = 3 \cdot (-2) + (2 + 2 + 2 + 2 + 2) = 4, \qquad g(X) = 3\,.$$

(Another argument for the ramification indices: X has an automorphism of order 3 sending y to ζy with $\zeta \neq 1$ and $\zeta^3 = 1$. This shows that at a point of X we have ramification 1 or 3.)

3. i) To show exactness we have to show exactness of the sequence of stalks at any point x. For any U kernel of $\overline{\partial}$ on $\mathcal{E}(U)$ are the holomorphic functions $O_X(U)$, hence also $O_{X,x}$ is the kernel of the induced map on the stalk. This shows the exactness at place $\mathcal{E}$. By the Dolbeault lemma a $(0,1)$-form is locally of the form $\overline{\partial} f\, d\overline{z}$ for some $f \in \mathcal{E}(U)$ in a open neighborhood of x. This proves the surjectivity $\mathcal{E}_x \to \mathcal{E}_x^{0,1}$. This proves i).

For ii) we apply the long exact cohomology sequence and get

$$0 \to H^0(X, O_X) \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{E}^{0,1}) \to H^1(X, O_X) \to H^1(X, \mathcal{E}) = (0),$$

where the last zero is derived from $\mathcal{E}$ being fine. Hence the map

$$\mathcal{E}^{0,1}(X) = H^0(X, \mathcal{E}^{0,1}) \to H^1(X, O_X)$$

is surjective with kernel $\overline{\partial}\mathcal{E}(X)$. We thus get the Dolbeault isomorphism $H^1(X, O_X) \cong \mathcal{E}^{0,1}(X)/\overline{\partial}\mathcal{E}(X)$.

4. We shall use that $H^0(X, O_X(D)) = (0)$ if $\deg D < 0$. Indeed, if $\deg D < 0$ then $H^0(X, O_X(D)) = (0)$ because a non-zero section f satisfies $\mathrm{div}(f) + D \geq 0$, but by $\deg(f) + \deg D = \deg D$ this has negative degree. So $f = 0$.

By Serre duality we have $\dim H^1(X, O_X(D)) = \dim H^0(X, O_X(K - D))$ and since $\deg(K - D) = 2g - 2 - \deg D < 0$ for $\deg D \geq 3$ we have $\dim H^1(X, O_X(D)) = 0$ for $\deg D \geq 3$. From Riemann-Roch we get

$$\dim H^0(X, O_X(D)) - \dim H^1(X, O_X(D)) = \deg D + 1 - 2 = \deg D - 1,$$

hence $h^0(X, O(D)) = \deg D - 1$.

Since X has genus 2 we have $H^1(X, O_X) = 2$. By Serre duality this shows that $H^0(X, O(K)) = 2$.

We also use that $H^0(X, O_X) = \mathbb{C}$ and $H^0(X, O_X(D)) = (0)$ if $\deg(D) = 0$ and $D \not\sim 0$. (Indeed, if $0 \neq f \in H^0(X, O(D))$ with $\deg D = 0$ then $(f) + D \geq 0$ and has degree 0, hence is zero, hence $(f) = -D \sim 0$.) For $\deg D = 2$ we have $h^1(X, O(D)) = h^0(X, O_X(K - D)) \neq 0$ if and only if $K - D \sim 0$. So R-R gives for $h^0(X, O_X(D))$ the value 2 if $D \sim K$ or 1 if $D \not\sim K$.

5. i) By the Hurwitz-Zeuthen formula we have $2g(X) - 2 = 2 \cdot (-2) + r$, hence $r = 2g(X) + 2$. ii) Suppose that $\phi(P_i) = Q_i \in \mathbb{P}^1$. On $\mathbb{P}^1$ we have $Q_i \sim Q_j$. (Change coordinates so that $P_i = 0$ and $P_j = \infty$ if $i \neq j$ and use the function z.) Hence by applying pullback ϕ^* we get $2P_i = \phi^*(Q_i) \sim \phi^*(Q_j) = 2P_j$. iii) The function field of X is a degree 2 extension of the function field $\mathbb{C}(x)$ of $\mathbb{P}^1$, say $\mathbb{C}(x)(y)$ with $y^2 = f(x)$ with f a rational function, say $f = a/b$ with a, b polynomials in x. Replacing y by yb we may assume that$f$ is a polynomial. Assume that P_1 lies over $\infty \in \mathbb{P}^1$. Then ϕ is ramified at the zeros and poles of f; say f has a pole at $Q_1 = \infty$ and simple zeros at Q_i with $i \neq 1$. The divisor of y is then $\sum_{i=2}^r P_i - (r-1)P_1$. Hence $\sum_{i=2}^r P_i \sim (r-1)P_1$. Adding P_1 to both sides gives iii).

For iv) take the meromorphic 1-form $\omega = dz$ on $\mathbb{P}^1$ with divisor $-2Q_1$. Then the divisor of $\phi^*(\omega)$ is $-3P_1 + \sum_{i=2}^r P_i$. (Note that at $Q_1 = \infty$ we have local coordinate $w = 1/z$ and $\omega = -dw/w^2$ so the pullback $\phi^*(\omega)$ in term of a local coordinate u with $u^2 = w$ equals $-d(u^2)/u^4 = -2du/u^3$.) Then we have $-3P_1 + \sum_{i=2}^r P_i \sim -4P_1 + \sum_{i=1}^r P_i \sim (2g-2)P_1$ by iii).

6. i) If $f \in H^0(X, O_X(P))$ is non-constant, then f has exactly one pole and one zero. The function f defines an isomorphism $X \to \mathbb{P}^1$ contradicting the assumption that $g > 0$. So $H^0(X, O_X(P)) = \mathbb{C}$. By Riemann-Roch we then have $h^0(P) - h^1(P) = 2 - g$, hence $h^1(P) = g - 1$. For ii) we consider the exact sequence of sheaves

$$0 \to O_X(n-1)P) \to O_X(nP) \to \mathcal{F} \to 0$$

with $\mathcal{F}$ a skyscraper sheaf with support at P. Taking the beginning of the long exact cohomology sequence we get

$$\begin{aligned} 0 \to & H^0(X, O_X((n-1)P) \to H^0(X, O_X(nP)) \xrightarrow{\rho} \mathbb{C} \to \\ & H^1(X, O_X((n-1)P)) \to H^1(X, O_X(nP)) \to 0 \end{aligned}$$

and depending on whether the map ρ is surjective or not we have

$$h^0((nP) = h^0((n-1)P) + 1 \quad \text{or} \quad h^0(nP) = h^0((n-1)P).$$

For iii) observe by that we by the same long exact sequence we have

$$h^1((nP) = h^1((n-1)P) - 1 \quad \text{or} \quad h^1(nP) = h^1((n-1)P).$$

and

$$h^0(nP) = h^0((n-1)P) + 1 \Longleftrightarrow h^1(nP) = h^1((n-1)P) \qquad (*)$$

The desired equality of iii) holds for $N = 1$. If N changes to $N + 1$ and the left hand side goes up by 1 then by (*) we see $h^1(NP) = h^1((N+1)P)$, so the right hand side goes also up by 1. For iv) we observe that $h^1(NP) = h^0(K - NP)$ and $\deg(K - NP) = 2g - 2 - N < 0$, hence $h^0(K - NP) = 0$. For v) remark that for $N \geq 2g - 1$ the right hand side of iii) is $N - g$. So there are g steps where $h^0(nP) = h^0((n-1)P)$; this means that every meromorphic function $f \in H^0(X, O_X(nP))$ has a pole of order at most $n - 1$ at P and no other poles. Now i) implies $n_1 = 1$ and iv) implies $n_g < 2g$.