# Exam Algebraic Geometry: Solutions

**June 10, 2015**

Please do five of the following six exercises. If you do all six the five best solutions will be counted. In the following $k$ denotes an algebraically closed field.

1)

i) Give the definition of a rational map of algebraic varieties.

ii) Show that the variety $X$ in $\mathbb{A}^n$ defined by $x_1x_2\cdots x_n = 1$ is birationally equivalent with $\mathbb{A}^{n-1}$.

iii) Give an example of two birationally equivalent affine varieties of dimension 3 that are not isomorphic.

Solution:

i) See Definition 5.13.

ii) Let $U \subset \mathbb{A}^{n-1}$ be the open subset given by $x_1x_2\cdots x_{n-1} \neq 0$. Let $\phi\colon U \to X$ be the morphism

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 1/(x_1x_2\cdots x_{n-1}))$$

and let $\psi\colon X \to U$ be the morphism

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}).$$

Then $\phi$ and $\psi$ are each other's inverses. Hence $X$ and $\mathbb{A}^{n-1}$ are birationally equivalent.

iii) Let $Y = \mathbb{A}^3$ and $Z = (\mathbb{A}^1 - \{0\}) \times \mathbb{A}^2$. Then $Y$ and $Z$ are affine varieties of dimension 3 (note that $Z$ is the open subset $D(x)$ of $Y$, cf. Prop. 2.20). Clearly, they are birationally equivalent. The coordinate ring of $Y$ is $k[x, y, z]$, in which no non-constant element has an inverse. But $x$ has an inverse in the coordinate ring of $Z$, so $Y$ and $Z$ are not isomorphic.

2) Let $X$ be a variety and $P \in X$ a point of $X$.

i) Give the definition of the local ring $\mathcal{O}_{X,P}$ of $P$.

ii) Assume that $X$ is a curve. Show: $P$ is a non-singular point of $X$ if and only if $\mathcal{O}_{X,P}$ is a discrete valuation ring.

Solution:

i) Cf. Definition 2.5, Def. 4.7, Exa. 4.8 (i), Def. 5.4, Def. 5.9.

ii) Cf. Theorem A4.20 and Corollary A4.21.

3) We assume that the characteristic of $k$ is 0. Let $X$ be the projective plane curve given by the homogeneous equation

$$(y^2 + xz)^2 + y^3 z = 0.$$

i) Determine the singular points of $X$.
ii) Show that the map from $\mathbb{P}^1$ to $\mathbb{P}^2$ given by

$$(s : t) \mapsto (st^3 - t^4 : -s^2t^2 : s^4)$$

defines a morphism with $X$ as its image.
iii) Show that the induced map from $\mathbb{P}^1$ to $X$ is the normalization of $X$ in its function field.
iv) For each singular point $P$ of $X$, find an element contained in the normalization of $\mathcal{O}_{X,P}$, but not in $\mathcal{O}_{X,P}$ itself.

Solution:
i) Cf. 7.16: we can use the projective tangent space of $X$ at $P$; put $F = (y^2+xz)^2+y^3z$, then the singular points are given by the simultaneous vanishing of $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$; this implies that $F$ vanishes as well (Euler's identity; char. 0). Now $\partial F/\partial x = 2(y^2 + xz)z$. Note that $z = 0$ implies $y^2 + xz = 0$. So $y^2 + xz = 0$ is forced; then $y^3z = 0$, so $y = 0$ or $z = 0$. This gives the two points $(1 : 0 : 0)$ and $(0 : 0 : 1)$ in $\mathbb{P}^2$.
ii) The map is clearly a morphism from $\mathbb{P}^1$ to $\mathbb{P}^2$, since the 3 polynomials $st^3 - t^4$, $-s^2t^2$, and $s^4$ are of the same degree and don't vanish simultaneously on $\mathbb{P}^1$. We also verify that the image is contained in $X$, since $y^2+xz = s^5t^3$ and $y^3z = -s^{10}t^6$. The line $z = 0$ is not contained in $X$, so the open part $z \neq 0$ is dense and has affine equation $(y^2 + x)^2 + y^3 = 0$ after setting $z = 1$ (which we may). This is clearly an irreducible curve: after applying the isomorphism $(x, y) \mapsto (x + y^2, y)$ the equation becomes $x^2+y^3 = 0$, the standard cuspidal curve, image of $\mathbb{A}^1$ under $t \mapsto (t^3, -t^2)$, hence irreducible (cf. Exa. 2.24). So $X$ is irreducible as well, hence the nonconstant map is surjective.
iii) Certainly, $\mathbb{P}^1$ is normal, and the map is finite and onto. But we also need to check that it induces a birational equivalence. This, however, follows from what was written above: setting $s = 1$, the map becomes $t \mapsto (t^3 - t^4, -t^2)$, exactly the map we used to prove that $X$ is irreducible (via the standard cuspidal curve, birationally equivalent to $\mathbb{A}^1$).
iv) On the standard cuspidal curve above, $w = x/y$ is such an element, satisfying the monic equation $w^2 + y = 0$. Similarly, on $(y^2 + x)^2 + y^3 = 0$ we have the element $w = (y^2 + x)/y$ satisfying $w^2 + y = 0$; $w$ is in the normalization of the local ring at $(0 : 0 : 1)$, but not in the local ring itself. At the other singular point $(1 : 0 : 0)$, we may set $x = 1$ and obtain the affine equation $(y^2+z)^2+y^3z = 0$. Now $w = (y^2 + z)/y$ does what we want: it satisfies $w^2 + yz = 0$, but is not contained in the local ring, since $z/y$ isn't.

4) We assume that $k$ has characteristic 0. Let $X \subset \mathbb{A}^4$ be the algebraic subset given by the equations

$$xy - z^2 = x^2w^3 - y^6 = 0.$$

In this exercise we will show that $X$ is irreducible.

i) Explain why every irreducible component of $X$ has dimension at least 2.

ii) Show that the open subset $U$ of $X$ given by $x \neq 0$ is dense in $X$.

iii) Show that $U$ is isomorphic to an open subset of the hypersurface $Y \subset \mathbb{A}^3$ (with coordinates $(x, z, w)$) given by the equation

$$x^8w^3 - z^{12} = 0.$$

iv) Show that the morphism from $\mathbb{A}^2$ to $Y$ given by

$$(a, b) \mapsto (a^3, a^2b, b^4)$$

is onto. Conclude that $Y$ is irreducible. Conclude that $X$ is irreducible as well.

Solution:

i) We are intersecting 2 hypersurfaces. Each irreducible component of a hypersurface has codimension 1 (Prop. 6.8 (ii)). Applying the affine dimension theorem Prop. 6.10 (i) finishes the proof.

ii) In a variety, every nonempty open subset is dense. But we don't know yet that $X$ is a variety; we need that $X$ is irreducible for that, which is what we are trying to prove. Concretely, we need to check that the open subset $U$ doesn't miss an entire irreducible component. The complement of $U$ is given by $x = 0$, which (on $X$) implies $z = 0$ and $y = 0$. This is just the affine line (the $w$-axis), of dimension 1, hence (by (i)) not a component. This proves that $U$ is dense in $X$.

iii) On $U$, we can write $y = z^2/x$, giving the equation $x^2w^3 - z^{12}/x^6 = 0$, or equivalently (as long as $x \neq 0$) $x^8w^3 - z^{12} = 0$. Clearly, $U$ is then isomorphic with the open subset $x \neq 0$ of the hypersurface $x^8w^3 - z^{12} = 0$ in $\mathbb{A}^3$ with coordinates $(x, z, w)$. (As above, this open subset is dense.)

iv) We can clearly find $a$ with $a^3 = x$ and $b$ with $b^4 = w$. Note that $a$ is unique up to a third root of 1, and $b$ up to a fourth root of 1 (popular notation like $\sqrt[3]{x}$ tends to obscure this!). Then $z^{12} = a^{24}b^{12}$, so $z = \zeta a^2 b$, where $\zeta$ is a 12th root of 1. Since $a^2$ is also a third root of 1, we finish the first step by noting that any 12th root of 1 can be written as a product of a third root of 1 and a fourth root of 1: the morphism is onto. Since $\mathbb{A}^2$ is irreducible, $Y$ is irreducible as well (from the definitions). Then $U$ is irreducible and hence its closure $X$ as well.

5) Let $G = \mathrm{Grass}(2,4) = \mathbb{G}(1,\mathbb{P}^3)$ be the Grassmannian of lines in $\mathbb{P}^3$.

i) Let $\ell$ be a given line in $\mathbb{P}^3$. Show that there exists a natural subvariety $G_\ell$ of $G$ corresponding to the lines in $\mathbb{P}^3$ that meet $\ell$. Determine the dimension of this subvariety.

Recall that $G$ can be given as a subvariety of $\mathbb{P}^5$ (with coordinates $(X_{01} : X_{02} : X_{03} : X_{12} : X_{13} : X_{23})$), namely, the hypersurface with equation

$$X_{01}X_{23} - X_{02}X_{13} + X_{03}X_{12} = 0.$$

The line in $\mathbb{P}^3$ through $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$ corresponds to the point

$$(a_0b_1 - a_1b_0 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0 : a_1b_2 - a_2b_1 : a_1b_3 - a_3b_1 : a_2b_3 - a_3b_2).$$

ii) Let $\ell$ be the line through $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. Show that $G_\ell$ is given by intersecting $G$ with a linear subspace of $\mathbb{P}^5$.

iii) Argue why there should be two lines meeting four given lines in general position.

Solution:

i) It is a closed condition for a line in $\mathbb{P}^3$ to meet $\ell$. So there is a closed subset $G_\ell$ of $G$ corresponding to such lines. To see that $G_\ell$ is irreducible, we may argue as follows. There exists a natural morphism from $\mathbb{P}^3 \times \mathbb{P}^3 \setminus \Delta$ (the complement of the diagonal) onto $G$: it sends a pair of distinct points in $\mathbb{P}^3$ to the point of $G$ corresponding to the unique line through the two points. The fiber over a point of $G$ corresponding to a line consists of the pairs of distinct points on the line. This recomputes the dimension of $G$ as $6 - 2 = 4$. Similarly, there exists a natural morphism from $\ell \times \mathbb{P}^3 \setminus \Delta(\ell)$ onto $G_\ell$: it sends a pair of distinct points (at least one on $\ell$) to the unique line through the points. It follows that $G_\ell$ is irreducible. The fiber over any point of $G_\ell$ not corresponding to $\ell$ itself consists (essentially) of the points on the corresponding line that don't lie on $\ell$. We can conclude that the dimension of $G_\ell$ is 3. Alternatively, consider a plane $P$ containing $\ell$. Any line in $\mathbb{P}^3$ meets $P$ (proj. dim. thm.). The lines that meet $P$ in a point of $\ell$ form a closed subset of $G$ of codimension 1.

ii) We see that $\ell$ corresponds to the point $(0 : 0 : 0 : 0 : 0 : 1)$ of $G \subset \mathbb{P}^5$. A general point of $\ell$ has coordinates $(0 : 0 : p : q)$. The only (Plücker) coordinate guaranteed to vanish for a line through such a point is $X_{01}$. Conversely, if $a_0b_1 - a_1b_0 = 0$, then the line through $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$ meets $\ell$. Thus, $G_\ell$ is given by intersecting $G$ with the hyperplane $X_{01} = 0$.

iii) Let $p$, $q$, $r$, and $s$ be four lines in general position. Each of $G_p$, $G_q$, $G_r$, and $G_s$ is given by intersecting $G$ with a hyperplane (say $H_p, \ldots, H_s$). The lines meeting $p$, $q$, $r$, and $s$ correspond to the intersection of $G_p$, $G_q$, $G_r$, and $G_s$, i.e., the intersection of $G$ with the intersection of $H_p$, $H_q$, $H_r$, and $H_s$, i.e., the intersection of $G$ with a line in $\mathbb{P}^5$. But $G$ itself is a quadric hypersurface; intersecting it with a line, we expect two points of intersection (which could coincide, however; an extra argument is required to exclude this).

6) Let $X$ be an irreducible complete non-singular curve and let $D = \sum_P n_P P$ be a divisor on $X$. Define for an open set $U$ of $X$

$$\mathcal{L}(D)(U) = \{f \in k(X)^* : \mathrm{div}(f|U) + D_{|U} \geq 0\} \cup \{0\},$$

where $D_{|U} = \sum_{P \in U} n_P P$ is the restriction of $D$ to $U$.

i) Show that $\mathcal{L}(D)(U)$ is a $\mathcal{O}(U)$-module.

ii) Show that $U \mapsto \mathcal{L}(D)(U)$ defines a sheaf on $X$.

iii) Show that $\mathcal{L}(D)(X) = (0)$ if $\deg D < 0$.

iv) Show that $\mathcal{L}(D)(X)$ is a $k$-vector space of dimension $\leq \deg D + 1$.

Solution:

i) It is clearly a sub-$k$ vector space of $k(X)$ (cf. Divisors, 1.7 (i)), hence an abelian group. We can multiply with regular functions on $U$ since $\mathrm{div}(g) \geq 0$ (as divisor on $U$) for $g \in \mathcal{O}(U)$.

ii) For each open $U$ we have an abelian group $\mathcal{L}(D)(U)$; we also have natural restriction maps, so we have a presheaf. (As usual, the value on $\emptyset$ is $\{0\}$.) We need to check the sheaf axiom. Let $U$ be a nonempty open and $\{U_\alpha\}_{\alpha \in I}$ an open cover. An element of $k(X)$ is determined by its restriction to any nonempty $U_\alpha$, so uniqueness is obvious. The existence of a rational function on $U$ is also clear. Its divisor on $U$ fulfils the desired inequality: we need to check the inequality at every point of $U$, but every point is contained in some $U_\alpha$.

iii) Divisors, Prop. 1.7 (iii).

iv) Divisors, Prop. 1.7 (iv).