Exam Riemann Surfaces (2013)

January 8, 2014. 2-5 pm, Location: VU:WN-Q112

Solve five of the following six problems. If you solve all of these the five best solutions will be counted.

1. Let Y be a non-empty open subset of a Riemann surface X. Let $\omega \in \mathcal{E}^{1,0}(Y)$ be a C^∞-form of type $(1,0)$. Prove: ω is holomorphic if and only if ω is closed.

2. Let X be a compact Riemann surface that is a degree 3 cover of the Riemann sphere $\mathbb{P}^1$ given by $y^3 = f(x)$ with $f \in \mathbb{C}[x]$ a polynomial of degree 4 with distinct zeros. Determine the genus of X.

3. Let X be a compact Riemann surface. Let $\mathcal{E}$ (resp. $\mathcal{E}^{0,1}$) be the sheaf of C^∞-functions (resp. C^∞-differential forms of type $(0,1)$) on X and $\mathcal{O}_X \subset \mathcal{E}$ the subsheaf of holomorphic functions.

i) Show that the following sequence of sheaves is exact: $0 \to \mathcal{O}_X \to \mathcal{E} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \to 0\,.$

ii) Prove the isomorphism $H^1(X, \mathcal{O}_X) \cong \mathcal{E}^{0,1}(X)/\overline{\partial}\,\mathcal{E}(X)$.

4. Let X be a compact Riemann surface of genus 2 and D a divisor on X. Show that

$$\dim H^0(X, \mathcal{O}_X(D)) = \begin{cases} \deg(D) - 1 & \deg(D) \geq 3 \\ 2 & D \sim K, \text{ a canonical divisor} \\ 1 & \deg D = 2, D \not\sim K \\ 0 & \deg D < 0. \end{cases}$$

5. Let X be a compact hyperelliptic Riemann surface of genus $g > 1$ and let $\phi : X \to \mathbb{P}^1$ be the non-constant holomorphic map of degree 2. Let $P_1, \ldots, P_r$ be the ramification points of ϕ on X.

i) Prove that $r = 2g + 2$.

ii) Prove the linear equivalence: $2P_i \sim 2P_j$ for $1 \leq i, j \leq 2g + 2$.

iii) Prove the linear equivalence: $\sum_{i=1}^{2g+2} P_i \sim (2g + 2)\, P_1$.

iv) Prove the linear equivalence $K \sim (2g - 2)P_1$ with K a canonical divisor of X.

6. Let X be a compact Riemann surface of genus $g > 0$ and P a point of X. We write $h^i(nP) = \dim H^i(X, \mathcal{O}_X(nP))$ for $i = 0, 1$.

i) Show that $h^0(P) = 1$ and $h^1(P) = g - 1$.

ii) Show that for $n > 0$ one has: $h^0(nP) - h^0((n-1)P)$ is either 0 or 1.

iii) Show by induction on N that

$$\#\{n : 1 \leq n \leq N \quad \text{such that } h^0(nP) - h^0((n-1)P) = 1\} = N - g + h^1(NP).$$

iv) Prove that $h^1(NP) = 0$ for $N > 2g - 2$.

v) Conclude that there are exactly g integers $n_1 < n_2 < \cdots < n_g$ such that there exists no meromorphic function on X having a pole of order n_i at P and no other poles. Furthermore, $n_1 = 1$ and $n_g < 2g$.