

ON VECTOR-VALUED SIEGEL MODULAR FORMS  
OF DEGREE 2 AND WEIGHT  $(j, 2)$

(WITH TWO APPENDICES BY GAËTAN CHENEVIER)

FABIEN CLÉRY AND GERARD VAN DER GEER

Received: October 10, 2017

Revised: April 26, 2018

Communicated by Don Blasius

ABSTRACT. We formulate a conjecture that describes the vector-valued Siegel modular forms of degree 2 and level 2 of weight  $\mathrm{Sym}^j \otimes \det^2$  and provide some evidence for it. We construct such modular forms of weight  $(j, 2)$  via covariants of binary sextics and calculate their Fourier expansions illustrating the effectivity of the approach via covariants. Two appendices contain related results of Chenevier; in particular a proof of the fact that every modular form of degree 2 and level 2 and weight  $(j, 1)$  vanishes.

2010 Mathematics Subject Classification: 11F46, 11F70, 14J15

Keywords and Phrases: Siegel modular forms, small weight

1. INTRODUCTION

The usual methods for determining the dimensions of spaces of Siegel modular forms do not work for low weights. For Siegel modular forms of degree 2 this means that we do not have formulas for the dimensions of the spaces of Siegel modular forms of weight  $(j, k)$ , that is, corresponding to  $\mathrm{Sym}^j \otimes \det^k$ , in case  $k < 3$ . In this paper we propose a description of the spaces of cusp forms of weight  $(j, 2)$  on the level 2 principal congruence subgroup

$$\Gamma_2[2] = \ker(\mathrm{Sp}(4, \mathbb{Z}) \rightarrow \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z}))$$

of  $\Gamma_2 = \mathrm{Sp}(4, \mathbb{Z})$  and we provide some evidence for this conjectural description. Let  $S_{j,k}(\Gamma_2[2])$  be the space of cusp forms of weight  $(j, k)$ , that is, corresponding to the factor of automorphy  $\mathrm{Sym}^j(c\tau + d) \det(c\tau + d)^k$  on the group  $\Gamma_2[2]$ . Recall that the group  $\mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$  is isomorphic to the symmetric group  $\mathfrak{S}_6$ . We fix an explicit isomorphism by identifying the symplectic lattice over  $\mathbb{Z}/2\mathbb{Z}$  with the subspace  $\{(a_1, \dots, a_6) \in (\mathbb{Z}/2\mathbb{Z})^6 : \sum a_i = 0\}$  modulo the diagonally embedded

$\mathbb{Z}/2\mathbb{Z}$  with form  $\sum_i a_i b_i$  as in [3, Section 2]; it is given explicitly on generators of  $\mathfrak{S}_6$  in [10, Section 3, (3.2)]. So  $\mathfrak{S}_6$  acts on the space of cusp forms  $S_{j,k}(\Gamma_2[2])$  and this space thus decomposes into isotypical components for the symmetric group  $\mathfrak{S}_6$ . The irreducible representations of  $\mathfrak{S}_6$  correspond to the partitions of 6 and we thus have for each such partition  $\varpi$  a subspace  $S_{j,k}(\Gamma_2[2])^{s[\varpi]}$  of  $S_{j,k}(\Gamma_2[2])$  where  $\mathfrak{S}_6$  acts as  $s[\varpi]$ . Note that the case  $s[6]$  corresponds to cusp forms on  $\mathrm{Sp}(4, \mathbb{Z})$ , while the case  $s[1^6]$  corresponds to modular forms of weight  $(j, k)$  on  $\mathrm{Sp}(4, \mathbb{Z})$  with a quadratic character:

$$S_{j,k}(\Gamma_2[2])^{s[1^6]} = S_{j,k}(\mathrm{Sp}(4, \mathbb{Z}), \epsilon)$$

with  $\epsilon$  the unique quadratic character of  $\mathrm{Sp}(4, \mathbb{Z})$ .

Before we formulate our conjecture we recall that the group  $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}) \cong \mathfrak{S}_3$  acts on the space  $S_k(\Gamma_1[2])$  of cusp forms on the principal congruence subgroup of level 2  $\Gamma_1[2] = \ker(\mathrm{SL}(2, \mathbb{Z}) \rightarrow \mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}))$ . We can thus decompose this space in isotypical components corresponding to the irreducible representations of  $\mathfrak{S}_3$ . The map  $f(z) \mapsto f(2z)$  defines an isomorphism  $S_k(\Gamma_1[2]) \xrightarrow{\sim} S_k(\Gamma_0(4))$  with  $\Gamma_0(4)$  the usual congruence subgroup of  $\Gamma_1 = \mathrm{SL}(2, \mathbb{Z})$ . If we write its isotypical decomposition as

$$S_k(\Gamma_1[2]) = a_k s[3] + b_k s[2, 1] + c_k s[1^3],$$

then  $\dim S_k(\Gamma_1) = a_k$ ,  $\dim S_k(\Gamma_0(2))^{\mathrm{new}} = b_k - a_k$  and  $\dim S_k(\Gamma_0(4))^{\mathrm{new}} = c_k$  with their generating series given by

$$\sum a_k t^k = t^{12}/(1 - t^4)(1 - t^6), \quad \sum b_k t^k = t^8/(1 - t^2)(1 - t^6),$$

and

$$\sum c_k t^k = t^6/(1 - t^4)(1 - t^6).$$

The Fricke involution  $w_2 : \tau \mapsto -1/2\tau$  defines an involution on  $S_k(\Gamma_0(2))^{\mathrm{new}}$  and this space splits into eigenspaces  $S_k^\pm(\Gamma_0(2))^{\mathrm{new}}$  and for  $k > 2$  we have

$$\dim S_k^+(\Gamma_0(2))^{\mathrm{new}} - \dim S_k^-(\Gamma_0(2))^{\mathrm{new}} = \begin{cases} -1 & k \equiv 2 \pmod{8} \\ 0 & k \equiv 4, 6 \pmod{8} \\ 1 & k \equiv 0 \pmod{8}. \end{cases}$$

We recall the notion of Yoshida type lifts. Yoshida lifts are explained in [27]; see also [28, 26, 5, 21]. These are eigen forms associated to a pair of elliptic modular eigenforms whose spinor L-function is a product of the twisted L-functions of the elliptic modular cusp forms. In [3] a number of conjectures on the existence of Yoshida lifts were made and these were proved by Rösner [20]. These conjectures deal with Siegel modular cusp forms of weight  $(j, k)$  with  $k \geq 3$ . It can be extended to the case of weight  $(j, 2)$ . We denote the subspace of  $S_{j,2}(\Gamma_2[2])^{s[\varpi]}$  generated by Yoshida lifts by  $YS_{j,2}^{s[\varpi]}$ .

**THEOREM 1.1.** *We have  $YS_{j,2}^{s[\varpi]} = 0$  unless we are in the following cases:*

- (1)  $\varpi = [1^6]$  and  $YS_{j,2}^{s[\varpi]}$  is generated by the  $Y(f, g)$  with  $f$  and  $g$  eigen-newforms of level  $\Gamma_0(2)$  of different sign. In this case we have

$$\dim YS_{j,2}^{s[\varpi]} = \dim S_{j+2}^+(\Gamma_0(2))^{\text{new}} \otimes S_{j+2}^-(\Gamma_0(2))^{\text{new}}.$$

- (2)  $\varpi = [2, 1^4]$  and  $YS_{j,2}^{s[\varpi]}$  is generated by the  $Y(f, g)$  with  $f$  and  $g$  non proportional eigen-newforms on  $\Gamma_0(4)$ . The multiplicity  $\mu(j)$  of  $s[2, 1^4]$  in  $YS_{j,2}^{s[\varpi]}$  is then

$$\mu(j) = \dim \Lambda^2 S_{j+2}(\Gamma_0(4))^{\text{new}}.$$

- (3)  $\varpi = [2^3]$  and  $YS_{j,2}^{s[\varpi]}$  is generated by the  $Y(f, g)$  with  $f$  and  $g$  non proportional eigen-newforms on  $\Gamma_0(2)$  with the same sign. The multiplicity  $\nu(j)$  of  $s[2^3]$  is

$$\nu(j) = \dim \Lambda^2 S_{j+2}^+(\Gamma_0(2))^{\text{new}} \oplus \Lambda^2 S_{j+2}^-(\Gamma_0(2))^{\text{new}}.$$

The proof of this theorem follows from results of Rösner and Weissauer, in a way very similar to Rösner’s proof of the Bergström-Faber-van der Geer conjecture in weight  $k \geq 3$  [20, §5.5]. In the second appendix Chenevier explains how to derive it.

We now formulate our conjecture.

CONJECTURE 1.2. *The space  $S_{j,2}(\Gamma_2[2])$  is generated by Yoshida type lifts.*

Note that this implies that  $S_{j,2}(\Gamma_2[2])^{s[\varpi]} = (0)$  unless  $\varpi = [1^6], [2, 1^4]$  or  $[2^3]$ . In particular, it implies that  $S_{j,2}(\Gamma_2) = (0)$ . The evidence we have for the latter is the following.

THEOREM 1.3. *We have  $\dim S_{j,2}(\Gamma_2) = 0$  for  $j \leq 52$ .*

For  $j \leq 20$  the vanishing of  $S_{j,2}(\Gamma_2)$  was proved by Ibukiyama, Wakatsuki and Uchida [16, Lemma 2.1], [17], and [25].

The evidence we have for the  $s[1^6]$ -part of the conjecture is the following.

THEOREM 1.4. *The dimension of  $S_{j,2}(\Gamma_2, \epsilon)$  is given by the coefficient of  $t^j$  in the expansion of  $t^{12}/(1 - t^6)(1 - t^8)(1 - t^{12})$  for  $j \leq 30$ .*

Modular forms in  $S_{j,2}(\Gamma_2, \epsilon)$  can be constructed explicitly using covariants as explained in [7]. We prove this theorem by constructing a basis of the space  $S_{j,7}(\Gamma_2)$  using covariants of binary sextics (see [7]) and then by checking which forms are divisible by the cusp form  $\chi_5 \in S_{0,5}(\Gamma_2, \epsilon)$ . We thus give generators for these spaces  $S_{j,2}(\Gamma_2[2])^{s[1^6]}$  for  $j \leq 30$  and we can calculate Hecke eigenvalues for these. For  $j > 30$  this becomes quite laborious.

For all irreducible representations we have the following vanishing result:

PROPOSITION 1.5. *For any  $\varpi$  we have  $\dim S_{j,2}(\Gamma_2[2])^{s[\varpi]} = 0$  for  $j < 12$ .*

We end with some remarks on other ‘small’ weights. The vanishing of  $S_{j,1}(\Gamma_2)$  follows from work of Skoruppa [23]. In an appendix to this paper besides providing a different proof of the vanishing of  $S_{j,2}(\Gamma_2)$  for  $j \leq 38$ , Chenevier gives a proof of the vanishing of  $S_{j,1}(\Gamma_2[2])$ . For  $k = 3$  one knows that  $S_{j,3}(\Gamma_2) = (0)$  for  $j < 36$ . But  $S_{36,3}(\Gamma_2)$  is 1-dimensional and using covariants we can construct a form of this weight in a relatively easy manner; cf. the remarks in [17, p. 207] on the difficulty of constructing such a form.

ACKNOWLEDGEMENT. We thank Gaëtan Chenevier warmly for asking the question on the existence of modular forms of weight  $(j, 2)$  on the full group  $\mathrm{Sp}(4, \mathbb{Z})$  and for agreeing to add his results in the form of an appendix. He also pointed out an error in an earlier version of our conjecture and provided a proof of the extension of Rösner’s result on Yoshida lifts. We thank Mirko Rösner for correspondence. We also thank the Max-Planck Institut für Mathematik in Bonn for excellent working conditions. The research of the first author was partly supported by the EPSRC grant EP/N031369/1.

## 2. MODULAR FORMS OF DEGREE TWO

For the definitions of Siegel modular forms and elementary properties we refer to [14]. We denote the Siegel upper half space of degree  $g$  by  $\mathfrak{H}_g$ . The Siegel modular group  $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$  acts on  $\mathfrak{H}_g$  by fractional linear transformations

$$\tau \mapsto (a\tau + b)(c\tau + d)^{-1} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}) \text{ and } \tau \in \mathfrak{H}_g.$$

If  $\rho : \mathrm{GL}(g, \mathbb{C}) \rightarrow \mathrm{GL}(W)$  is a finite-dimensional complex representation then a holomorphic map  $f : \mathfrak{H}_g \rightarrow W$  is called a Siegel modular form of weight  $\rho$  if  $f(a\tau + b)(c\tau + d)^{-1} = \rho(c\tau + d)f(\tau)$  for all  $(a, b; c, d) \in \Gamma_g$ . The space of modular forms of weight  $\rho$  is finite-dimensional and denoted by  $M_\rho(\Gamma_g)$ .

If  $g = 2$  then an irreducible representation of  $\mathrm{GL}(2, \mathbb{C})$  is of the form  $\mathrm{Sym}^j(\mathrm{St}) \otimes \det(\mathrm{St})^k$  with  $\mathrm{St}$  the standard representation of  $\mathrm{GL}(2, \mathbb{C})$  for some  $j \in \mathbb{Z}_{\geq 0}$  and  $k \in \mathbb{Z}$ .

For  $\rho = \mathrm{Sym}^j(\mathrm{St}) \otimes \det(\mathrm{St})^k$  we denote  $M_\rho$  by  $M_{j,k}$  and we call  $(j, k)$  the weight. If  $j = 0$  we are dealing with scalar-valued modular forms. The space of Siegel modular forms of degree 2 and weight  $(j, k)$  is denoted by  $M_{j,k}(\Gamma_2)$ .

There is the Siegel operator  $\Phi_g$  that maps Siegel modular forms of degree  $g$  to Siegel modular forms of degree  $g - 1$ . The kernel of  $\Phi_2$  in  $M_{j,k}(\Gamma_2)$  is called the space of cusp forms of weight  $(j, k)$  and denoted by  $S_{j,k}(\Gamma_2)$ . Note that for  $k = 2$  we have  $M_{j,2}(\Gamma_2) = S_{j,2}(\Gamma_2)$ , see [16, Lemma 2.1].

For a finite index subgroup  $\Gamma$  of  $\mathrm{Sp}(4, \mathbb{Z})$  we have similar notions. Here we deal with the groups  $\Gamma_2$  and  $\Gamma_2[2]$ . The quotient group  $\Gamma_2/\Gamma_2[2] \cong \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$  is identified with the symmetric group  $\mathfrak{S}_6$  as in the Introduction. This group acts in a natural way on the space of cusp forms  $S_{j,k}(\Gamma_2[2])$  and we can decompose this space in isotypical components  $S_{j,k}(\Gamma_2[2])^{s[\varpi]}$  corresponding to the irreducible representations  $s[\varpi]$  of  $\mathfrak{S}_6$  which in turn correspond bijectively to the partitions  $\varpi$  of 6.

The ring  $R$  of scalar-valued Siegel modular forms on  $\Gamma_2$  was determined by Igusa in the 1960s, see [18]. In the 1980s Tsushima gave in [24] formulas for the dimensions of the spaces of vector-valued cusp forms on a subgroup between  $\Gamma_2[2]$  and  $\Gamma_2$ . Bergström extended this to  $\Gamma_2[2]$  with the action of  $\mathfrak{S}_6$ , see [2]. We thus know the dimension of  $S_{j,k}(\Gamma_2[2])^{s[\varpi]}$  for all  $j$  and  $k \geq 3$ . The vector-valued modular forms of degree 2 form a ring  $M = \bigoplus_{j,k} M_{j,k}(\Gamma_2)$ . It is also a module over the ring  $R$ . For level 2 similar things hold. A vector-valued Siegel modular form  $f$  of weight  $(j, k)$  on  $\Gamma_2$  has a Fourier-Jacobi expansion

$$f(\tau) = \sum_{m \geq 0} \varphi_m(\tau_1, z) e^{2\pi i m \tau_2} \quad \text{where } \tau = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix}$$

with  $\varphi_m : \mathfrak{H}_1 \times \mathbb{C} \rightarrow \text{Sym}^j(\mathbb{C}^2)$  a holomorphic map that satisfies certain functional equations under the action of the so-called Jacobi group, and this is of the form  $\text{SL}(2, \mathbb{Z}) \times H(\mathbb{Z})$  with  $H(\mathbb{Z})$  a Heisenberg group. This group is embedded in  $\Gamma_2$  via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\lambda, \mu, \nu) \mapsto \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \nu \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with action

$$\tau \mapsto \begin{pmatrix} (a\tau_1 + b)/(c\tau_1 + d) & z/(c\tau_1 + d) \\ z/(c\tau_1 + d) & \tau_2 - cz^2/(c\tau_1 + d) \end{pmatrix}$$

and

$$\tau \mapsto \begin{pmatrix} \tau_1 & z + \lambda\tau_1 + \mu \\ z + \lambda\tau_1 + \mu & \tau_2 + \lambda^2\tau_1 + 2\lambda z + \lambda\mu + \nu \end{pmatrix}.$$

The fact that  $f$  is a modular form of weight  $(j, k)$  implies the corresponding functional equations

$$\varphi_m\left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{z}{c\tau_1 + d}\right) e^{-2\pi i m \frac{cz^2}{c\tau_1 + d}} = \text{Sym}^j \begin{pmatrix} c\tau_1 + d & cz \\ 0 & 1 \end{pmatrix} (c\tau_1 + d)^k \varphi_m(\tau_1, z)$$

and

$$\varphi_m(\tau_1, z + \lambda\tau_1 + \mu) e^{2\pi i m (\lambda^2\tau_1 + 2\lambda z + \lambda\mu)} = \text{Sym}^j \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \varphi_m(\tau_1, z),$$

where we write  $\varphi_m$  as the transpose of the row vector  $(\varphi_m^{(0)}, \dots, \varphi_m^{(j)})$ .

**COROLLARY 2.1.** *If  $f \in M_{j,k}(\Gamma_2)$  (resp.  $f \in S_{j,k}(\Gamma_2)$ ) then the last coordinate  $\varphi_m^{(j)}$  of the coefficient  $\varphi_m$  of  $e^{2\pi i m \tau_2}$  in the Fourier-Jacobi expansion of  $f$  is a Jacobi form (resp. Jacobi cusp form) of weight  $k$  and index  $m$ .*

We note that Jacobi cusp forms of weight 2 and index  $m$  are zero for  $m < 37$ , see [11, pp. 117-120]. This imposes strong conditions on forms of weight  $(j, 2)$  on  $\Gamma_2$ .

Since we shall compute the action of Hecke operators later we now describe formulas for the action of Hecke operators on forms on  $\Gamma_2$ . For forms without character we refer to [9, Appendix], so we deal with the case of forms with a character. For  $f \in M_{j,k}(\Gamma_2, \epsilon)$  we write its Fourier expansion as

$$f(\tau) = \sum_{n \geq 0} a(n) e^{\pi i \text{Tr}(n\tau)},$$

where  $n$  runs over the positive semi-definite half-integral symmetric matrices. We will write  $[n_1, n_2, n_3]$  for  $\begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}$ . For an odd prime  $p$  we denote by  $T_p$  the Hecke operator for  $\Gamma_2$  at  $p$ . Then we write the transform of  $f$  under  $T_p$  as

$$T_p(f)(\tau) = \sum_{n \geq 0} a_p(n) e^{\pi i \text{Tr}(n\tau)}.$$

Here for  $p \not\equiv 1 \pmod 3$  and  $p \neq 3$  the coefficient  $a_p([1, 1, 1])$  is given by  $a([p, p, p])$ , and for  $p = 3$  by

$$a([3, 3, 3]) - 3^{k-2} \text{Sym}^j \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} a([1, 3, 3]),$$

while for  $p \equiv 1 \pmod 3$  by

$$a([p, p, p]) + p^{k-2} \sum_{i=1}^2 (-1)^{m_i} \text{Sym}^j \begin{pmatrix} p & -m_i \\ 0 & 1 \end{pmatrix} a\left(\left[\frac{1+m_i+m_i^2}{p}, 1+2m_i, p\right]\right),$$

where in the latter case  $m_1$  and  $m_2$  are the two roots of the polynomial  $1 + X + X^2$  over  $\mathbb{F}_p$ , which we view here as the set  $\{0, \dots, p-1\}$ . Similarly, the coefficient  $a_{p^2}([1, 1, 1])$  of the transform of  $f$  under the Hecke operator  $T_{p^2}$  is given for  $p \not\equiv 1 \pmod 3$  and  $p \neq 3$  by  $a([p^2, p^2, p^2])$ , and for  $p = 3$  by

$$a([9, 9, 9]) - 3^{k-2} \text{Sym}^j \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} a([3, 9, 9]).$$

As an example, consider the modular form

$$\chi_5 \in S_{0,5}(\Gamma_2, \epsilon),$$

the product of the ten even theta characteristics and the square root of Igusa’s cusp form  $\chi_{10}$ , that will play an important role in this paper. It provides a check on these formulas for the Hecke operators. Indeed, one knows

$$\lambda_p(\chi_5) = p^3 + a_p(f) + p^4, \quad \lambda_{p^2}(\chi_5) = \lambda_p(\chi_5)^2 - (p^4 + p^3)\lambda_p(\chi_5) + p^8,$$

where  $f = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + \dots$  is the normalized Hecke eigenform in  $S_8^+(\Gamma_0(2))^{\text{new}}$ , which illustrates that  $\chi_5$  is a Saito-Kurokawa lift. One can check that the above formulas agree with this.

### 3. RESTRICTION TO THE DIAGONAL

In order to put restrictions on the existence of Siegel modular forms we restrict these to the ‘diagonal’ given by the embedding

$$i : \mathfrak{H}_1 \times \mathfrak{H}_1 \rightarrow \mathfrak{H}_2, \quad (z_1, z_2) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

The stabilizer of  $i(\mathfrak{H}_1 \times \mathfrak{H}_1)$  in  $\mathrm{Sp}(4, \mathbb{R})$  is an extension by  $\mathbb{Z}/2\mathbb{Z}$  of the image of  $\mathrm{SL}(2, \mathbb{R})^2$  under the embedding

$$\left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

The extension by  $\mathbb{Z}/2\mathbb{Z}$  corresponds to the involution that interchanges  $\tau_1$  and  $\tau_2$  in

$$\tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} \in \mathfrak{H}_2$$

(and  $z_1$  and  $z_2$  on  $\mathfrak{H}_1^2$ ). This corresponds to the element  $\iota = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  in  $\Gamma_2$  with  $a = d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The stabilizer inside  $\Gamma_2$  (resp. inside  $\Gamma_2[2]$ ) is an extension by  $\mathbb{Z}/2\mathbb{Z}$  of  $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$  (resp. of  $\Gamma_1[2] \times \Gamma_1[2]$ ).

If  $F = (F_0, \dots, F_j)^t$  is a Siegel modular form of weight  $(j, k)$  of level 2, then its pullback under  $i$  to  $\mathfrak{H}_1 \times \mathfrak{H}_1$  gives rise to an element of  $(f_0, \dots, f_j)^t$  with  $f_l \in M_{j+k-l}(\Gamma_1[2]) \otimes M_{k+l}(\Gamma_1[2])$ .

By restricting a cusp form of level 1 we get cusp forms of level 1. The action of  $\iota$  is given by a map

$$S_{j+k-i}(\Gamma_1[2]) \otimes S_{k+i}(\Gamma_1[2]) \rightarrow S_{k+i}(\Gamma_1[2]) \otimes S_{j+k-i}(\Gamma_1[2]), \quad a \otimes b \mapsto (-1)^k b \otimes a$$

for  $\Gamma_1[2]$  and a similar one for  $\Gamma_1$ . So for a form of level 1 without character we loose no information by looking at

$$\bigoplus_{i=0}^{j/2-1} S_{j+k-i}(\Gamma_1) \otimes S_{k+i}(\Gamma_1) \bigoplus \begin{cases} \wedge^2 S_{j/2+k}(\Gamma_1) & \text{for } k \text{ odd} \\ \mathrm{Sym}^2 S_{j/2+k}(\Gamma_1) & \text{for } k \text{ even.} \end{cases}$$

By multiplying with  $\chi_5$  we get an injective map  $S_{j,2}(\Gamma_2, \epsilon) \rightarrow S_{j,7}(\Gamma_2)$ . The generating series for the dimensions is

$$\sum_{j=2}^{\infty} \dim S_{j,7}(\Gamma_2) t^j = \frac{t^{12}}{(1-t)(1-t^3)(1-t^4)(1-t^6)}.$$

We observe that our conjecture on  $S_{j,2}(\Gamma_2, \epsilon)$  implies that

$$\dim S_{j,7}(\Gamma_2) - \dim S_{j,2}(\Gamma_2, \epsilon) =$$

$$\sum_{i=0}^{j/2-1} \dim S_{j+7-i}(\Gamma_1) \dim S_{7+i}(\Gamma_1) + \dim \wedge^2 S_{j/2+7}(\Gamma_1),$$

or equivalently, that the restriction  $\rho$  to the diagonal fits in an exact sequence

$$0 \rightarrow S_{j,2}(\Gamma_2, \epsilon) \xrightarrow{\chi_5} S_{j,7}(\Gamma_2) \xrightarrow{\rho} \bigoplus_{i=0}^{j/2-1} S_{j+7-i}(\Gamma_1) \otimes S_{7+i}(\Gamma_1) \oplus \wedge^2 S_{j/2+7}(\Gamma_1) \rightarrow 0.$$

If  $F \in S_{j,k}(\Gamma_2, \epsilon)$  we find by using  $\iota$  that we can restrict to

$$\bigoplus_{i=0}^{j/2-1} S_{j+k-i}(\Gamma_1[2])^{s[1^3]} \otimes S_{k+i}(\Gamma_1[2])^{s[1^3]} \bigoplus \mathrm{Sym}^2(S_{j/2+k}(\Gamma_1[2])^{s[1^3]}).$$

Indeed, the group  $\mathfrak{S}_3 = \mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$  acts on  $S_k(\Gamma_1[2])$  and for a form on  $\Gamma_2$  with a character the components  $f_i, f'_i$  of the restriction to  $i(\mathfrak{H}_1 \times \mathfrak{H}_1)$  are modular forms on  $\Gamma_1[2]$  with a character, i.e. they lie in the  $s[1^3]$ -isotypical part

of  $S_k(\Gamma_1[2])$ . The module  $\oplus_k S_k(\Gamma_1[2])^{s[1^3]}$  is a module over the ring  $\mathbb{C}[e_4, e_6]$  of modular forms on  $\Gamma_1$  and is generated by the cusp form  $\delta = \eta^{12}$ , a square root of  $\Delta \in S_{12}(\Gamma_1)$ .

The generating series for the dimensions is now

$$\sum_{j=2}^{\infty} \dim S_{j,7}(\Gamma_2, \epsilon) t^j = \frac{t^6}{(1-t)(1-t^3)(1-t^4)(1-t^6)}.$$

Conjecturally we now find an exact sequence

$$0 \rightarrow S_{j,7}(\Gamma_2, \epsilon) \xrightarrow{\rho} \oplus_{i=0}^{\frac{j}{2}-1} S_{j+7-i}(\Gamma_0(4))^n \otimes S_{7+i}(\Gamma_0(4))^n \oplus \text{Sym}^2(S_{j/2+7}(\Gamma_0(4))^n) \rightarrow K \rightarrow 0,$$

where  $S_k(\Gamma_0(4))^n = S_k(\Gamma_0(4))^{\text{new}}$  and the dimension of the cokernel  $K$  can now be predicted by considering the algorithms used in [3] to calculate the dimension of  $S_{j,k}(\Gamma_2)$ . The extrapolation to  $k = 2$  of the algorithm gives negative numbers and one takes the negative of the outcome of the algorithm.

#### 4. CONSTRUCTING MODULAR FORMS USING COVARIANTS

In the paper [7] we explained how to use invariant theory to construct Siegel modular forms. In this paper we shall make extensive use of the procedure.

Let  $V$  be the standard representation space of  $\text{GL}(2, \mathbb{C})$  with basis  $x_1, x_2$ . We consider the space  $\text{Sym}^6(V)$  of binary sextics, where we write an element as

$$f = \sum_{i=0}^6 a_i \binom{6}{i} x_1^{6-i} x_2^i.$$

Sometimes we call this expression the universal binary sextic. For a description of invariants and covariants for the action of  $\text{GL}(2, \mathbb{C})$  we refer to [7, Section 3]. An invariant can be viewed as a polynomial in the coefficients  $a_i$  that is invariant under the action of  $\text{SL}(2, \mathbb{C})$ , while a covariant of degree  $(a, b)$  can be viewed as a form of degree  $a$  in the  $a_i$  and degree  $b$  in  $x_1, x_2$ . If  $A[\lambda_1, \lambda_2]$  is an irreducible representation of highest weight  $(\lambda_1 \geq \lambda_2)$  of  $\text{GL}(2, \mathbb{C})$  embedded equivariantly in  $\text{Sym}^d(\text{Sym}^6(V))$  this defines a covariant of degree  $(d, \lambda_1 - \lambda_2)$  and it is unique up to a multiplicative non-zero constant.

We denote the ring of covariants by  $\mathcal{C}$ . Clebsch and others constructed in the 19th century generators for this ring. There are 26 generators, 5 invariants and 21 covariants, satisfying many relations. They can be found in the book of Grace and Young [15, p. 156]. For the convenience of the reader we reproduce these here. In the following table  $C_{a,b}$  denotes a generator of degree  $(a, b)$ .



$a \setminus b$	0	2	4	6	8	10	12
1				$C_{1,6}$			
2	$C_{2,0}$		$C_{2,4}$		$C_{2,8}$		
3		$C_{3,2}$		$C_{3,6}$	$C_{3,8}$		$C_{3,12}$
4	$C_{4,0}$		$C_{4,4}$	$C_{4,6}$		$C_{4,10}$	
5		$C_{5,2}$	$C_{5,4}$		$C_{5,8}$		
6	$C_{6,0}$			$C_{6,6}^{(1)}$ $C_{6,6}^{(2)}$			
7		$C_{7,2}$	$C_{7,4}$				
8		$C_{8,2}$					
9			$C_{9,4}$				
10	$C_{10,0}$	$C_{10,2}$					
12		$C_{12,2}$					
15	$C_{15,0}$						

A theorem of Gordan says that all these covariants can be constructed explicitly by using so-called transvectants from the universal binary sextic. If  $\text{Sym}^m(V)$  denotes the space of binary quantics of degree  $m$  then we define the  $k$ th transvectant as follows. It is a map  $\text{Sym}^m(V) \times \text{Sym}^n(V) \rightarrow \text{Sym}^{m+n-2k}(V)$  that sends a pair  $(f, g)$  to

$$(f, g)_k = \frac{(m-k)!(n-k)!}{m!n!} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{\partial^k f}{\partial x_1^{k-j} \partial x_2^j} \frac{\partial^k g}{\partial x_1^j \partial x_2^{k-j}}.$$

When  $k = 1$ , we omit the index:  $(f, g) = (f, g)_1$ . The next table gives the construction of the covariants in the preceding table.

1	$C_{1,6} = f$			
2	$C_{2,0} = (f, f)_6$	$C_{2,4} = (f, f)_4$	$C_{2,8} = (f, f)_2$	
3	$C_{3,2} = (C_{1,6}, C_{2,4})_4$	$C_{3,6} = (f, C_{2,4})_2$	$C_{3,8} = (f, C_{2,4})$	$C_{3,12} = (f, C_{2,8})$
4	$C_{4,0} = (C_{2,4}, C_{2,4})_4$	$C_{4,4} = (f, C_{3,2})_2$	$C_{4,6} = (f, C_{3,2})$	$C_{4,10} = (C_{2,8}, C_{2,4})$
5	$C_{5,2} = (C_{2,4}, C_{3,2})_2$	$C_{5,4} = (C_{2,4}, C_{3,2})$	$C_{5,8} = (C_{2,8}, C_{3,2})$	
6	$C_{6,0} = (C_{3,2}, C_{3,2})_2$	$C_{6,6}^{(1)} = (C_{3,6}, C_{3,2})$	$C_{6,6}^{(2)} = (C_{3,8}, C_{3,2})_2$	
7	$C_{7,2} = (f, C_{3,2}^2)_4$	$C_{7,4} = (f, C_{3,2}^2)_3$		
8	$C_{8,2} = (C_{2,4}, C_{3,2}^2)_3$			
9	$C_{9,4} = (C_{3,8}, C_{3,2}^2)_4$			
10	$C_{10,0} = (f, C_{3,2}^3)_6$	$C_{10,2} = (f, C_{3,2}^3)_5$		
12	$C_{12,2} = (C_{3,8}, C_{3,2}^3)_6$			
15	$C_{15,0} = (C_{3,8}, C_{3,2}^4)_8$			

Let  $M$  be the ring of vector-valued Siegel modular forms of degree 2. It is a module over the ring  $R$  of scalar-valued Siegel modular forms of degree 2. In [7] we defined maps

$$M \longrightarrow \mathcal{C} \xrightarrow{\nu} M_{\chi_{10}},$$

where  $M_{\chi_{10}}$  is the localization of  $M$  at  $\chi_{10}$ . A modular form of weight  $(j, k)$  maps to a covariant of degree  $(j/2 + k, j)$  and a covariant of degree  $(a, b)$  is sent to a meromorphic modular form of weight  $(b, a - b/2)$ . Under the map  $\nu$  the universal binary sextic  $f$  is mapped to  $\chi_{6,3}/\chi_5$  of weight  $(6, -2)$ . Here  $\chi_{6,3}$

is a holomorphic form in  $S_{6,3}(\Gamma_2, \epsilon)$ . The beginning of its Fourier expansion is given in formula (4) of [7].

In practice instead of  $\nu$  often we use a slightly modified map

$$\mu : \mathcal{C} \longrightarrow M \oplus M_\epsilon,$$

where  $M_\epsilon = \oplus M_{j,k}(\Gamma_2, \epsilon)$  is the  $R$ -module of modular forms with a character. Under  $\mu$  the universal sextic  $f$  is mapped to  $\chi_{6,3}$ . Then a covariant maps of degree  $(a, b)$  maps to a holomorphic Siegel modular form of weight  $(b, 6a - b/2)$  and character  $\epsilon^a$ .

*Remark 4.1.* Since  $\chi_{6,3}$  vanishes simply at infinity the definition of  $\mu$  implies that the image under  $\mu$  of a covariant of degree  $(a, b)$  vanishes at infinity with order  $\geq a$ . Recall that the order of vanishing of  $\chi_5$  at infinity is 1.

For example, Igusa's generators  $E_4, E_6, \chi_{10}, \chi_{12}$  and  $\chi_{35}$  of  $R$  are up to a non-zero multiplicative constant obtained as

$$\begin{aligned} E_4 &= \mu(75 C_{4,0} - 8 C_{2,0}^2) / \chi_{10}^2, \\ E_6 &= \mu(224 C_{2,0}^3 - 1425 C_{2,0} C_{4,0} - 1125 C_{6,0}) / \chi_{10}^3 \\ \chi_{10}^6 &= \mu(C_{\chi_{10}}), \quad \chi_{12} = \mu(C_{2,0}), \quad \chi_{35} = \mu(C_{15,0}) / \chi_5^{11}, \end{aligned}$$

with  $C_{\chi_{10}}$ , up to a multiplicative constant equal to the discriminant, given by

$$\begin{aligned} &768 C_{2,0}^5 - 7625 C_{4,0} C_{2,0}^3 \\ &- 1875 (7 C_{6,0} C_{2,0}^2 - 10 C_{4,0}^2 C_{2,0} - 30 C_{6,0} C_{4,0} - 13860 C_{10,0}) . \end{aligned}$$

*Remark 4.2.* The first scalar-valued cusp form on  $\Gamma_2$  with character is of weight 30 and can be obtained by dividing  $\mu(C_{15,0})$  by  $\chi_5^{12}$ . Note that we have  $M_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2, \epsilon)$ , (see [17, p. 198]).

## 5. CUSP FORMS OF WEIGHT $(j, 2)$ ON $\Gamma_2$ WITH A CHARACTER

Our conjecture says that  $S_{j,2}(\Gamma_2, \epsilon) = (0)$  for  $j < 12$ . We begin by showing this.

LEMMA 5.1. *For  $j = 0, 2, 4, 6, 8$  and  $10$ , we have  $S_{j,2}(\Gamma_2, \epsilon) = (0)$ .*

*Proof.* We know that  $\dim S_{2j,4}(\Gamma_2) = 0$  for  $j = 0, 2, 4, 6, 8, 10$ . Assume that for one of these values of  $j$  there is a non-zero element  $f \in S_{j,2}(\Gamma_2, \epsilon)$ . Then  $\text{Sym}^2(f) \in S_{2j,4}(\Gamma_2) = (0)$  must be zero. Using the fact that the ring of holomorphic functions on  $\mathfrak{H}_2$  is an integral domain, we get a contradiction.  $\square$

The first case where  $S_{j,2}(\Gamma_2, \epsilon)$  is predicted to be non-zero is for  $j = 12$ . In the paper [7] we constructed a modular form  $\chi_{12,2}$  in this space using the covariant  $C_{3,12}$  associated to  $A[15, 3]$  occurring in  $\text{Sym}^3(\text{Sym}^6(V))$ , after dividing by the

cuspidal form  $\chi_{10}$ . Its Fourier expansion starts with

$$\chi_{12,2}(\tau) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2(R-R^{-1}) \\ 9(R+R^{-1}) \\ 12(R-R^{-1}) \\ 0 \\ -12(R-R^{-1}) \\ -9(R+R^{-1}) \\ -2(R-R^{-1}) \\ 0 \\ 0 \\ 0 \end{pmatrix} Q_1 Q_2 + \dots,$$

where  $Q_1 = e^{\pi i \tau_1}$ ,  $Q_2 = e^{\pi i \tau_2}$  and  $R = e^{\pi i \tau_{12}}$  for  $\tau = (\frac{\tau_1}{\tau_{12}} \ \frac{\tau_{12}}{\tau_2})$ . By multiplication by  $\chi_{6,3}$  we get an injective map  $S_{12,2}(\Gamma_2, \epsilon) \rightarrow S_{18,5}(\Gamma_2)$  and this latter space is 1-dimensional.

**COROLLARY 5.2.** *We have  $\dim S_{12,2}(\Gamma_2, \epsilon) = 1$  and it is generated by  $\chi_{12,2}$ .*

We compute a few Hecke eigenvalues as described in Section 2. To compute these Hecke eigenvalues, we used the following Fourier coefficients:

$$\begin{aligned} a([1, 1, 1])^t &= [0, 0, 0, 2, 9, 12, 0, -12, -9, -2, 0, 0, 0] \\ a([1, 3, 3])^t &= [0, 0, 0, -2, -27, -156, -504, -996, -1233, -934, -396, -72, 0] \\ a([3, 3, 3])^t &= [0, 216, 1188, 258, -7749, -12708, 0, 12708, 7749, -258, -1188, -216, 0] \\ a([5, 5, 5])^t &= [0, 0, 0, -106920, -481140, -641520, 0, 641520, 481140, 106920, 0, 0, 0] \\ a([1, 5, 7])^t &= [0, 0, 0, 2, 45, 444, 2520, 9060, 21375, 33046, 32220, 17928, 4320] \\ a([7, 7, 7])^t &= [0, -8208, -45144, -542204, -2101338, -2711496, 0, 2711496, 2101338, \\ &\quad 542204, 45144, 8208, 0] \\ a([3, 9, 7])^t &= [0, -72, -1188, -8854, -39339, -115764, -236880, -343884, \\ &\quad -354141, -253514, -120132, -33912, -4320] \end{aligned}$$

These and a few more (too big to be written here) yield the following eigenvalues:

$p$	3	5	7	11	13	17
$\lambda_p$	-600	-53460	-369200	4084344	-2845700	131681700

together with  $\lambda_9 = -1090791$ . We find that for the operator  $T_p$  these are indeed of the form  $\lambda_p(f^+) + \lambda_p(f^-)$  with  $f^\pm$  generators of  $S_{14}^\pm(\Gamma_0(2))^{\text{new}}$ , with

$$\begin{aligned} f^+(\tau) &= q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + 117504q^6 + \dots \\ f^-(\tau) &= q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + 79104q^6 + \dots, \end{aligned}$$

while for  $T_{p^2}$  we find  $\lambda_p(f^+)^2 + \lambda_p(f^+)\lambda_p(f^-) + \lambda_p(f^-)^2 - 2(p+1)p^j$ . This fits with being a Yoshida lift.

**LEMMA 5.3.** *We have  $S_{14,2}(\Gamma_2, \epsilon) = (0) = S_{16,2}(\Gamma_2, \epsilon)$ .*

*Proof.* To see that  $S_{14,2}(\Gamma_2, \epsilon) = (0)$  we multiply a form in  $S_{14,2}(\Gamma_2, \epsilon)$  with  $\chi_5$  and we end up in  $S_{14,7}(\Gamma_2)$  and this space is generated by a form associated to  $C_{1,6}C_{3,8}$  after division by  $\chi_5^2$ . Restricting this form  $\chi_{14,7}$  to  $\mathfrak{H}_1^2$  gives  $\sum_{i=0}^7 (f_i \otimes f'_i)$  and only the term  $f_5 \otimes f'_5$  in  $S_{16}(\Gamma_1) \otimes S_{12}(\Gamma_1)$  can be non-zero and it is equal to  $56 e_4 \Delta \otimes \Delta$ . So it does not vanish along  $\mathfrak{H}_1 \times \mathfrak{H}_1$ , hence  $\chi_{14,7}$  is not divisible by  $\chi_5$ . We conclude  $S_{14,2}(\Gamma_2[2], \epsilon) = (0)$ . For  $S_{16,2}(\Gamma_2, \epsilon)$  we multiply by  $\chi_{6,3}$  and land in  $S_{22,5}(\Gamma_2)$  and this space is zero.  $\square$

Next we deal with the case of weight  $(18, 2)$ .

PROPOSITION 5.4. *The space  $S_{18,2}(\Gamma_2, \epsilon)$  has dimension 1.*

*Proof.* First we construct a non-zero element in this space by using the covariant

$$C = 135 C_{1,6}^2 C_{4,6} + 56 C_{1,6} C_{2,0} C_{3,12} - 270 C_{2,8} C_{4,10} - 930 C_{3,6} C_{3,12}.$$

It occurs in  $\text{Sym}^6(\text{Sym}^6(V))$  and provides a cusp form,  $F_C$ , of weight  $(18, 27)$  on  $\Gamma_2$ . The order of vanishing of  $F_C$  along  $\mathfrak{H}_1 \times \mathfrak{H}_1$  is 5, so we can divide it by  $\chi_5^5$  and we get by Remark 4.1 a cusp form, denoted  $\chi_{18,2}$ , of weight  $(18, 2)$  on  $\Gamma_2$  with character.

Again we multiply by  $\chi_5$  and land in  $S_{18,7}(\Gamma_2)$ . This space is 2-dimensional and we can construct a basis using the following covariants

$$C_1 = C_{3,12}(8 C_{1,6} C_{2,0} - 75 C_{3,6}), C_2 = C_{1,6}^2 C_{4,6} - 2 C_{2,8} C_{4,10} - 3 C_{3,6} C_{3,12}.$$

They occur in  $\text{Sym}^6(\text{Sym}^6(V))$  and provide two cusp forms,  $F_{C_i}$ , of weight  $(18, 27)$  on  $\Gamma_2$ . Each cusp form  $F_{C_i}$  vanishes with order 4 along  $\mathfrak{H}_1 \times \mathfrak{H}_1$ , so we can divide it by  $\chi_5^4$  and get a cusp form,  $\chi_{18,7}^{(i)}$ , of weight  $(18, 7)$  on  $\Gamma_2$ . The cusp forms  $\chi_{18,7}^{(1)}$  and  $\chi_{18,7}^{(2)}$  are  $\mathbb{C}$ -linearly independent as can be read off from the first terms of their Fourier expansions and the pullbacks to  $\mathfrak{H}_1 \times \mathfrak{H}_1$  are of the form  $\sum_{r=0}^9 f_r \otimes f'_r$  with only non-zero terms for  $r = 5$  and these are  $216 e_4^2 \Delta \otimes \Delta$  and  $48 e_4^2 \Delta \otimes \Delta$ . Up to a non-zero scalar there is only one non-trivial linear combination, that vanishes along  $\mathfrak{H}_1 \times \mathfrak{H}_1$  and that gives a non-zero form in  $S_{18,2}(\Gamma_2, \epsilon)$  after division by  $\chi_5$ .  $\square$

PROPOSITION 5.5. *The space  $S_{20,2}(\Gamma_2, \epsilon)$  has dimension 1.*

*Proof.* We construct a non-zero form in this space by taking the covariant

$$C = 224 C_{1,6}^2 C_{5,8} + 312 C_{1,6} C_{2,4} C_{4,10} - 560 C_{1,6} C_{2,8} C_{4,6} - 108 C_{1,6} C_{3,6} C_{3,8} + 728 C_{2,0} C_{2,8} C_{3,12} - 1235 C_{2,4}^2 C_{3,12}$$

occurring in  $\text{Sym}^7(\text{Sym}^6(V))$  and providing a cusp form,  $F_C$ , of weight  $(20, 32)$  on  $\Gamma_2$  with character. The order of vanishing of  $F_C$  along  $\mathfrak{H}_1 \times \mathfrak{H}_1$  is 6, so we can divide it by  $\chi_5^6$  and we get a cusp form,  $\chi_{20,2}$ , of weight  $(20, 2)$  on  $\Gamma_2$  with character.

In a similar way we construct a basis of the space  $S_{20,7}(\Gamma_2)$  by taking the covariants

$$\begin{aligned} C_1 &= 480 C_{1,6}^2 C_{5,8} - 180 C_{1,6} C_{3,6} C_{3,8} + 728 C_{2,0} C_{2,8} C_{3,12} - 1315 C_{2,4}^2 C_{3,12}, \\ C_2 &= 80 C_{1,6}^2 C_{5,8} - 80 C_{1,6} C_{2,8} C_{4,6} + 104 C_{2,0} C_{2,8} C_{3,12} - 125 C_{2,4}^2 C_{3,12}, \\ C_3 &= 80 C_{1,6}^2 C_{5,8} - 40 C_{1,6} C_{2,4} C_{4,10} + 56 C_{2,0} C_{2,8} C_{3,12} - 55 C_{2,4}^2 C_{3,12}. \end{aligned}$$

which provide cusp forms with character of weight  $(20, 32)$  and these are divisible by  $\chi_5^2$  and thus give cusp forms of weight  $(20, 7)$  generating  $S_{20,7}(\Gamma_2)$ . By restriction to the diagonal one sees that there is just a 1-dimensional space of forms vanishing on the diagonal.  $\square$

The case of weight  $(24, 2)$  is dealt with in a similar way.

PROPOSITION 5.6. *We have  $\dim S_{24,2}(\Gamma_2, \epsilon) = 2$ .*

*Proof.* We know that  $\dim S_{24,7}(\Gamma_2) = 5$  and we can construct a basis using the procedure described in Section 4. In the case at hand we have  $A[39, 15]$  occurring in  $\text{Sym}^9(\text{Sym}^6(V))$  with multiplicity 13 and this gives a subspace of  $S_{24,42}(\Gamma_2, \epsilon)$  of dimension 13. One checks that there is a 5-dimensional subspace of forms vanishing with multiplicity 7 along the diagonal and dividing by  $\chi_5^7$  gives a 5-dimensional subspace of  $S_{24,7}(\Gamma_2)$ , hence the whole space. Again one checks that there is a 2-dimensional space of forms vanishing on the diagonal and we can divide these forms by  $\chi_5$ . So the two generators of  $S_{24,2}(\Gamma_2, \epsilon)$  are defined by the covariants  $C_1$  and  $C_2$  given respectively by

$$\begin{aligned} & - 499408 C_{1,6}^2 C_{2,0}^2 C_{3,12} - 1505385 C_{1,6}^3 C_{6,6}^{(1)} - 14727825 C_{1,6}^2 C_{2,4} C_{5,8} \\ & 6916455 C_{1,6}^2 C_{2,8} C_{5,4} - 5728590 C_{1,6}^2 C_{3,12} C_{4,0} + 6972210 C_{1,6} C_{2,0} C_{2,8} C_{4,10} \\ & + 4257120 C_{1,6} C_{2,0} C_{3,6} C_{3,12} + 2182950 C_{2,8}^2 C_{5,8} + 11708550 C_{2,8} C_{3,6} C_{4,10} \\ & + 595350 C_{2,8} C_{3,12} C_{4,4} + 35171325 C_{3,6}^2 C_{3,12} - 400950 C_{3,8}^3 \end{aligned}$$

and

$$\begin{aligned} & - 42235648 C_{1,6}^2 C_{2,0}^2 C_{3,12} + 4434583545 C_{1,6}^3 C_{6,6}^{(1)} + 580982220 C_{1,6}^3 C_{6,6}^{(2)} \\ & + 4919972400 C_{1,6}^2 C_{2,4} C_{5,8} + 4827362400 C_{1,6}^2 C_{3,12} C_{4,0} \\ & - 3504891600 C_{1,6} C_{2,0} C_{2,8} C_{4,10} + 1245336960 C_{1,6} C_{2,0} C_{3,6} C_{3,12} \\ & - 4131252720 C_{2,8}^2 C_{5,8} - 24904998720 C_{2,8} C_{3,6} C_{4,10} \\ & - 281640240 C_{2,8} C_{3,12} C_{4,4} - 58751907480 C_{3,6}^2 C_{3,12} + 1375354080 C_{3,8}^3. \end{aligned}$$

$\square$

The order of vanishing of  $F_{C_i}$  along  $\mathfrak{H}_1 \times \mathfrak{H}_1$  is 8, so we can divide it by  $\chi_5^8$  and we get two cusp forms,  $\chi_{24,2}^{(i)}$ ,  $(i = 1, 2)$  of weight  $(24, 2)$  on  $\Gamma_2$  with character. We set  $\chi_{24,2}^{(1)} = -12150 F_{C_1} / \chi_5^8$  and  $\chi_{24,2}^{(2)} = -675(5368 F_{C_1} + 5 F_{C_2}) / 31528 \chi_5^8$ .

Then their Fourier expansions are given by

$$\chi_{24,2}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 104(R-R^{-1}) \\ 1092(R+R^{-1}) \\ 3640(R-R^{-1}) \\ 0 \\ -27678(R-R^{-1}) \\ -58905(R+R^{-1}) \\ -2916(R-R^{-1}) \\ 148470(R+R^{-1}) \\ 190778(R-R^{-1}) \\ 0 \\ \vdots \end{pmatrix} Q_1 Q_2 + \dots, \chi_{24,2}^{(2)}(\tau) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2(R-R^{-1}) \\ 17(R+R^{-1}) \\ 60(R-R^{-1}) \\ 110(R+R^{-1}) \\ 98(R-R^{-1}) \\ 0 \\ \vdots \end{pmatrix} Q_1 Q_2 + \dots,$$

where  $Q_1 = e^{i\pi\tau_1}$ ,  $Q_2 = e^{i\pi\tau_2}$ ,  $R = e^{i\pi\tau_{12}}$ . The action of  $\iota = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \Gamma_2$  with  $a = d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  implies that the  $i$ th coordinate is equal to  $(-1)^{k+1}$  times the  $(j + 1 - i)$ th coordinate, which gives the non-displayed coordinates. A Hecke eigenbasis of the space  $S_{24,2}(\Gamma_2, \epsilon)$  is:

$$F_1 = 439\chi_{24,2}^{(1)} + (114847 + 650\sqrt{106705}) \chi_{24,2}^{(2)}$$

$$F_2 = 439\chi_{24,2}^{(1)} + (114847 - 650\sqrt{106705}) \chi_{24,2}^{(2)}$$

with eigenvalues

$p$	$\lambda_p(F_1)$	$\lambda_{p^2}(F_1)$
3	$287880 - 4800\sqrt{106705}$	$545747143689 - 2293459200\sqrt{106705}$
5	$711981900 + 1555200\sqrt{106705}$	-
7	$-41070905840 + 92534400\sqrt{106705}$	-
11	$-10344705071976 + 4819953600\sqrt{106705}$	-

in perfect agreement with the eigenforms being Yoshida lifts. Indeed a basis of the space  $S_{26}(\Gamma_0[2])^{\text{new}}$  is given by

$$f = q - 4096 q^2 + 97956 q^3 + 16777216 q^4 + 341005350 q^5 - 401227776 q^6 + \dots$$

$$g = q + 4096 q^2 + (2048 - \frac{a}{2}) q^3 + 16777216 q^4 + (431848374 + 162 a) q^5 + \dots$$

$$g' = q + 4096 q^2 + (2048 + \frac{a}{2}) q^3 + 16777216 q^4 + (431848374 - 162 a) q^5 + \dots,$$

where  $a = -375752 + 9600\sqrt{106705}$ , and  $f, g' \in S_{26}^-$  and  $g \in S_{26}^+$ . Then we check for example that

$$\lambda_5(F_1) = 711981900 + 1555200\sqrt{106705} = a_5(f) + a_5(g)$$

$$\lambda_5(F_2) = 711981900 - 1555200\sqrt{106705} = a_5(f) + a_5(g').$$

PROPOSITION 5.7. *One has  $\dim S_{26,2}(\Gamma_2, \epsilon) = 1 = \dim S_{28,2}(\Gamma_2, \epsilon)$  and  $\dim S_{30,2}(\Gamma_2, \epsilon) = 2$ .*

*Proof.* The proof of this proposition is similar to the above. For the first statement we consider the space  $S_{26,7}(\Gamma_2)$  which has dimension 6 and construct a basis of this space using covariants associated to  $A[43, 17]$  in  $\text{Sym}^{10}(\text{Sym}^6(V))$

which occurs with multiplicity 17, thus giving rise to a 17-dimensional subspace of  $S_{26,47}(\Gamma_2)$ . By restricting along the diagonal one checks that there is a 6-dimensional subspace of cusp forms divisible by  $\chi_5^8$  leading to the construction of  $S_{26,7}(\Gamma_2)$ . Again by restricting to the diagonal one sees that there is exactly a 1-dimensional subspace of this space that vanish along the diagonal. By dividing by  $\chi_5$  we thus find the space  $S_{26,2}(\Gamma_2, \epsilon)$ .

For weight  $(28, 2)$  we now use the representation  $A[47, 19]$  that occurs with multiplicity 23 in  $\text{Sym}^{11}(\text{Sym}^6(V))$  and leading to a 23-dimensional subspace of  $S_{28,52}(\Gamma_2)$  in which we find a 7-dimensional subspace of forms divisible by  $\chi_5^9$  and division gives forms that generate  $S_{28,7}(\Gamma_2)$ . In this space the subspace of forms divisible by  $\chi_5$  is of dimension 1, proving our claim.

In the case of weight  $(30, 7)$  the 9-dimensional space  $S_{30,7}$  is constructed using covariants resulting from  $A[51, 21]$  that occurs with multiplicity 31 in  $\text{Sym}^{12}(\text{Sym}^6(V))$  leading to a space of dimension 31 of cusp forms of weight  $(30, 57)$ . There is a 9-dimensional subspace of cusp forms divisible by  $\chi_5^{10}$  and we thus generate  $S_{30,7}(\Gamma_2)$ . It turns out that there is a 2-dimensional subspace of forms divisible by  $\chi_5$  and this proves the claim.  $\square$

For all the cases treated we can check our construction by verifying that the Hecke eigenvalues for  $p = 3, 5, 7, 11, 13, 17$  agree with the forms being Yoshida lifts like we indicated for  $j = 12$  and  $j = 24$ .

In a forthcoming paper ([8]) we shall use the relation with covariants to describe modules of forms with a character.

## 6. MODULAR FORMS OF WEIGHT $(j, 2)$ ON $\Gamma_2$

In this section we explain how we checked that  $S_{j,2}(\Gamma_2) = (0)$  for  $j \leq 52$ . We begin with a simple lemma. Recall that we have maps  $M \rightarrow \mathcal{C} \xrightarrow{\mu} M$ .

**LEMMA 6.1.** *Let  $f \in M_{j,k}(\Gamma_2)$ . Then there exists a covariant  $c_f$  of degree  $(d, j)$  with  $d \leq j/2 + k$  such that  $f = \nu(c_f) = \mu(c_f)/\chi_5^d$ . If  $f$  is a cusp form then there is a covariant  $c'_f$  of degree  $\leq j/2 + k - 10$  such that  $f = \mu(c'_f)/\chi_5^r$  for some  $r$ .*

*Proof.* The first statement follows directly from [7]. If  $f$  is a cusp form then the covariant it defines vanishes on the discriminant locus. But then the covariant  $c_f$  is divisible by the discriminant, and  $\mu(c_f)$  by  $\chi_5^{d+2}$ .  $\square$

This makes it possible to check the existence of a non-zero form  $f \in S_{j,2}(\Gamma_2)$  by checking whether the forms of weight  $(j, 2 + 5d)$  provided via  $\mu$  by the non-zero covariants of degree  $(d, j)$  with  $d \leq j/2 - 8$  are divisible by  $\chi_5^d$ . We applied this for values of  $j \leq 52$  using the covariants of degree  $d \leq 18$ . For smaller values of  $j$  other methods of showing that  $S_{j,2}(\Gamma_2) = (0)$  are available. We sketch some methods below. In this way we checked that  $S_{j,2}(\Gamma_2) = (0)$  for  $j \leq 52$ .

Another method is to construct a basis of  $S_{j,7}(\Gamma_2, \epsilon)$  by using covariants. We then check the divisibility by  $\chi_5$  of elements in  $S_{j,7}(\Gamma_2, \epsilon)$  by restricting the modular forms in this space to the diagonal.

As an illustration we give the proof for the case  $j = 24$ . We construct a basis of the 9-dimensional space  $S_{24,7}(\Gamma_2, \epsilon)$ . For this we use the covariants associated to the  $A[54, 30]$ -isotypical component of  $\text{Sym}^{14}(\text{Sym}^6(V))$ . The representation  $A[54, 30]$  occurs with multiplicity 65 and leads to a 65-dimensional subspace of modular forms of weight  $(24, 72)$  on  $\Gamma_2$ . By restricting to  $\mathfrak{H}_1 \times \mathfrak{H}_1$  we can check that there exists a 9-dimensional subspace of cusp forms that are divisible by  $\chi_5^{13}$ . This leads to a basis of  $S_{24,7}(\Gamma_2, \epsilon)$ . We then check by restriction to  $\mathfrak{H}_1 \times \mathfrak{H}_1$  again that there is no non-trivial element in this space that is divisible by  $\chi_5$ . This proves the result for  $j = 24$ .

We carried this out for all the cases  $j \leq 52$  and thus proved Theorem 1.3.

Sometimes there are other and easier ways to eliminate cases. For example, by restricting a modular form of weight  $(j, 2)$  to the diagonal we get an element  $\sum_{i=0}^{j/2} f_i \otimes f'_i \in \oplus_{i=0}^{j/2} S_{j+2-i}(\Gamma_1) \otimes S_{2+i}(\Gamma_1)$ . If  $j < 24$ ,  $j \neq 20$  the spaces in question are zero. Therefore a form  $f \in S_{j,2}(\Gamma_2)$  will vanish on  $\mathfrak{H}_1 \times \mathfrak{H}_1$ . But then  $f/\chi_5$  will be a holomorphic modular form of weight  $(j, -3)$  and this has to be zero. So  $f = 0$  for  $j \leq 18$  and  $j = 22$ .

As yet another example of eliminating cases we give a somewhat different argument for  $j = 26$ . We write elements of  $F \in S_{j,k}(\Gamma_2)$  as vectors  $F = (F_0, \dots, F_j)^t$  with the  $F_i$  holomorphic functions on  $\mathcal{H}_2$ , that is, in a module of rank 27 over the ring  $\mathcal{F}$  of holomorphic functions on  $\mathcal{H}_2$ . Take a basis  $s_1, \dots, s_3$  of  $S_{26,6}(\Gamma_2)$  and a basis  $s_4, \dots, s_{12}$  of  $S_{26,8}(\Gamma_2)$ . If there exists a non-zero form  $f$  of weight  $(26, 2)$  then the vectors  $E_4 f$  and  $E_6 f$  are linearly dependent and thus the exterior product  $s_1 \wedge \dots \wedge s_{12}$  must vanish. By calculating bases of  $S_{26,6}(\Gamma_2)$  and  $S_{26,8}(\Gamma_2)$  one can check that this exterior product does not vanish. So  $S_{26,2}(\Gamma_2) = (0)$ .

## 7. OTHER SMALL WEIGHTS

We begin with an elementary argument that shows that  $S_{j,k}(\Gamma_2[2]) = (0)$  for  $j \leq 8$  and  $k \leq 2$ .

PROPOSITION 7.1. *For  $j \leq 8$  and  $k \leq 2$  we have  $\dim S_{j,k}(\Gamma_2[2]) = 0$ .*

*Proof.* We need to deal with the cases  $j$  even and  $k = 1$  and  $k = 2$  only since for other values  $S_{j,k}(\Gamma_2[2])$  vanishes. We restrict to the ten components of the Humbert surface  $H_1$  in  $\Gamma_2[2] \backslash \mathfrak{H}_2$ , one component of which is given by the diagonal  $\tau_{12} = 0$ . The group  $\mathfrak{S}_6$  acts transitively on these ten components. The stabilizer inside  $\mathfrak{S}_6$  of a component of  $H_1$  is an extension of  $\mathfrak{S}_3 \times \mathfrak{S}_3$  by  $\mathbb{Z}/2\mathbb{Z}$ .

By restricting a modular form  $f \in S_{j,1}(\Gamma_2[2])$  to a component we get an element of

$$\oplus_{r=0}^j S_{j+1-r}(\Gamma_1[2]) \otimes S_{1+r}(\Gamma_1[2])$$

and for  $f \in S_{j,2}(\Gamma_2[2])$  we get an element of

$$\oplus_{r=0}^j S_{j+2-r}(\Gamma_1[2]) \otimes S_{2+r}(\Gamma_1[2]).$$



For  $j \leq 8$  and  $k = 1$  and for  $j < 8$  and  $k = 2$  these spaces are zero. Thus a form  $f \in S_{j,2}(\Gamma_2[2])$  restricts to zero on all irreducible components of  $H_1$ , hence is divisible by  $\chi_5$ , and so  $f$  must be zero. For  $j = 8$  and  $k = 2$  the restriction to  $H_1$  gives an injective  $\mathfrak{S}_6$ -equivariant map

$$S_{8,2}(\Gamma_2[2]) \rightarrow \oplus_{i=1}^{10} \text{Sym}^2 S_6(\Gamma_1[2]),$$

where the action on the right is the induced representation from the extension of  $\mathfrak{S}_3 \times \mathfrak{S}_3$  by  $\mathbb{Z}/2\mathbb{Z}$  to  $\mathfrak{S}_6$ . Now  $S_6(\Gamma_1[2])$  is 1-dimensional and of type  $s[1^3]$  and we check that the representation of  $\mathfrak{S}_6$  on the 10-dimensional space  $\oplus_{i=1}^{10} \text{Sym}^2 S_6(\Gamma_1[2])$  is of type  $s[6]+s[4,2]$ . Since  $S_{8,2}(\Gamma_2) = (0)$ , we conclude that only  $S_{8,2}(\Gamma_2[2])^{s[4,2]}$  can be non-zero. If  $S_{8,2}(\Gamma_2[2])^{s[4,2]}$  is non-zero, then  $S_{8,2}(\Gamma_0[2])$  is non-zero (see [10, Section 9]). By restricting we get an element in a similar decomposition as before but with  $\Gamma_1[2]$  replaced by  $\Gamma_0[2]$ . As we know that all these spaces are zero, we can divide by  $\chi_5$ . This contradiction concludes our claim.  $\square$

More generally we have

PROPOSITION 7.2. For  $j < 12$  we have  $S_{j,2}(\Gamma_2[2]) = (0)$ .

*Proof.* The space  $S_{j,2}(\Gamma_2[2])$  is defined over  $\mathbb{Q}$ . All the cusps of  $\Gamma_2[2]$  are defined over  $\mathbb{Q}$  and the action of  $\mathfrak{S}_6$  is defined over  $\mathbb{Q}$ . The q-expansion principle says that a modular form in  $S_{j,k}(\Gamma_2[2])$  with  $k \geq 3$  is defined over  $\mathbb{Q}$  if its Fourier coefficients at all cusps are defined over  $\mathbb{Q}$ , see [19, Cor. 1.6.2, 1.12.2] and [13, p. 140]. We apply this to  $f\chi_{10}$  with  $f \in S_{j,2}(\Gamma_2[2])$  defined over  $\mathbb{Q}$  and we conclude that the Fourier coefficients of  $\sigma(f\chi_{10})$  with  $\sigma \in \mathfrak{S}_6$  are real, hence also those of  $\sigma(f)$  and if  $f \neq 0$  we find by looking at the ‘first’ non-zero term in a Fourier expansion that

$$\sum_{\sigma \in \mathfrak{S}_6} \sigma(f)^2$$

is non-zero and because of  $\sigma(f^2) = \sigma(f)^2$  also invariant under  $\mathfrak{S}_6$ . Thus it defines a non-zero element of  $S_{2j,4}(\Gamma_2)$ . So  $S_{j,2}(\Gamma_2[2]) \neq (0)$  implies  $S_{2j,4}(\Gamma_2) \neq (0)$ . But we know that  $S_{2j,4}(\Gamma_2) = (0)$  for  $j < 12$ .  $\square$

*Remark 7.3.* Note that the argument of the proof shows that our conjecture on the vanishing of  $S_{j,2}(\Gamma_2)$  for all  $j$  implies the vanishing of  $S_{j,1}(\Gamma_2[2])$  for all  $j$ .

In order to put our evidence for the vanishing of  $S_{j,2}(\Gamma_2)$  in perspective we show a small table that gives for each value of  $k$  the smallest  $j_0$  such that  $\dim S_{j_0,k}(\Gamma_2) \neq 0$ .

$k$	3	4	5	6	7	8
$j_0$	36	24	18	12	12	6

We can easily construct the generators of the corresponding spaces. In [9] we constructed a generator  $\chi_{6,3}$  of  $S_{6,3}(\Gamma_2, \epsilon)$  and above we gave the generator  $\chi_{12,2}$

of  $S_{12,2}(\Gamma_2, \epsilon)$ . The modular forms  $\chi_{12,2}^2$ ,  $\chi_{6,3}\chi_{12,2}$ ,  $\chi_{6,3}^2$ ,  $\chi_5\chi_{12,2}$  and  $\chi_5\chi_{6,3}$  give the generators for  $k = 4, \dots, 8$ . We end by constructing a generator of  $S_{36,3}(\Gamma_2)$ ; the non-vanishing of this space plays a role in the appendix. We look in  $\text{Sym}^{11}(\text{Sym}^6(V))$  and at  $A[51, 15]$  occuring with multiplicity 17 there. We find the covariant

$$297 C_{1,6}^2 C_{3,8}^3 - 8316 C_{1,6} C_{3,8} C_{3,12} C_{4,10} + 4116 C_{1,6} C_{3,12}^2 C_{4,6} \\ - 5488 C_{2,0} C_{3,12}^3 + 9030 C_{2,4} C_{3,8} C_{3,12}^2$$

giving a form  $f$  of weight  $(36, 48)$  that is divisible by  $\chi_5^9$  and  $f/\chi_5^9$  generates  $S_{36,3}(\Gamma_2)$ . As a check we note that the Fourier coefficient at  $n = [1, 1, 1]$  is of the form

$$[0, 0, 0, 0, 0, 0, 0, 32/6089428125, 464/6089428125, \dots]$$

and the coefficient at  $n = [5, 5, 5]$  is

$$[0, 0, 0, 0, 0, 0, 0, -8687121398144/81192375, -125963260273088/81192375, \dots]$$

giving the eigenvalue for the Hecke operator  $T_5$  as their ratio  $-20360440776900$ , in agreement with the value given in [2].

#### APPENDIX A. BY GAËTAN CHENEVIER

Let  $j$  and  $k$  be integers with  $j \geq 0$ , and  $\Gamma \subset \text{Sp}_4(\mathbb{Z})$  a congruence subgroup. Recall that  $S_{j,k}(\Gamma)$  denotes the space of cuspidal Siegel modular forms for the subgroup  $\Gamma$  with values in the representation  $\text{Sym}^j \otimes \det^k$  of  $\text{GL}_2(\mathbb{C})$ . We first consider the full Siegel modular group  $\Gamma_2 = \text{Sp}_4(\mathbb{Z})$  and provide alternative proofs of the following results :

PROPOSITION A.1. *We have  $S_{j,1}(\Gamma_2) = 0$  for any  $j$ , and  $S_{j,2}(\Gamma_2) = 0$  for  $j \leq 38$ .*

The vanishing of  $S_{j,2}(\Gamma_2)$  for all  $j \leq 52$  is also proved in this paper by Cléry and van der Geer (Theorem 1.3). The vanishing of  $S_{j,1}(\Gamma_2)$  was at least known to Ibukiyama, who asserts in [16, p. 54] that it is a consequence of the vanishing of all Jacobi forms of weight 1 for  $\text{SL}_2(\mathbb{Z})$  proven by Skoruppa [23, Satz 6.1]. Here we shall rather use automorphic representation theoretic methods. First we need to fix some notations and make some preliminary remarks. We denote by  $r : \text{Sp}_4(\mathbb{C}) \rightarrow \text{GL}_4(\mathbb{C})$  the tautological inclusion.

For a real reductive Lie group  $H$  we shall denote the infinitesimal character of the Harish-Chandra module  $U$  by  $\text{inf } U$ . In the case  $H = \text{GL}_n(\mathbb{R})$  (resp.  $H = \text{PGSp}_4(\mathbb{R})$ ), and following Harish-Chandra and Langlands,  $\text{inf } U$  may be viewed in a canonical way as a semisimple conjugacy class in the Lie algebra  $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{C})$  (resp.  $\mathfrak{h} = \mathfrak{sp}_4(\mathbb{C})$ ). In both cases we may and shall identify this conjugacy class with the multiset of its eigenvalues in the natural representation of  $\mathfrak{h}$ . We denote by  $W_{\mathbb{R}}$  the Weil group of  $\mathbb{R}$  (a certain extension of  $\mathbb{Z}/2\mathbb{Z}$  by  $\mathbb{C}^\times$ ), and for any integer  $w$  we define  $I_w$  as the 2-dimensional representation of  $W_{\mathbb{R}}$  induced from the unitary character  $z \mapsto (z/|z|)^w$  of  $\mathbb{C}^\times$ .

(a) As we have  $S_{j,k}(\Gamma_2) = 0$  for any odd  $j$  or for  $k \leq 0$ , we may once and for all assume  $j \equiv 0 \pmod 2$  and  $k > 0$ . As is well-known, for any such  $(j, k)$  there is an irreducible unitary Harish-Chandra module for  $\mathrm{PGSp}_4(\mathbb{R})$ , unique up to isomorphism, generated by a highest-weight vector of  $K$ -type  $\mathrm{Sym}^j \otimes \det^k$ . This module, that we shall denote by  $U_{j,k}$ , is a holomorphic discrete series for  $k \geq 3$ , a limit of holomorphic discrete series for  $k = 2$ , and non-tempered for  $k = 1$ ; it is *non-generic* in all cases. We have  $\mathrm{inf} U_{j,k} = \{ \frac{j+2k-3}{2}, \frac{j+1}{2}, -\frac{j+1}{2}, -\frac{j+2k-3}{2} \}$ . More precisely, if  $\varphi : \mathrm{W}_{\mathbb{R}} \rightarrow \mathrm{Sp}_4(\mathbb{C})$  denotes the Langlands parameter of  $U_{j,k}$ , then we have  $r \circ \varphi \simeq I_{j+2k-3} \oplus I_{j+1}$  for  $k > 1$ , and  $r \circ \varphi \simeq I_j \otimes |\cdot|^{1/2} \oplus I_j \otimes |\cdot|^{-1/2}$  for  $k = 1$  (see e.g. [22] for a survey of those properties, and the references therein).

The relevance of  $U_{j,k}$  here is that if  $\pi$  is a cuspidal automorphic representation of  $\mathrm{PGSp}_4$  over  $\mathbb{Q}$  generated by an element of  $S_{j,k}(\Gamma_2)$ , then the Archimedean component  $\pi_{\infty}$  of  $\pi$  is isomorphic to  $U_{j,k}$ . The other important property of  $\pi$  is that  $\pi_p$  is unramified for each prime  $p$  (i.e. admits non-zero invariants under  $\mathrm{PGSp}_4(\mathbb{Z}_p)$ ). As  $\mathrm{PGSp}_4$  is isomorphic to the split classical group  $\mathrm{SO}_5$  over  $\mathbb{Z}$ , we may apply Arthur’s theory [1] to such a  $\pi$ .

(b) One of the main results of Arthur [1, Thm. 1.5.2] associates to any discrete automorphic representation  $\pi$  of  $\mathrm{PGSp}_4$  over  $\mathbb{Q}$  a unique isobaric automorphic representation  $\pi^{\mathrm{GL}}$  of  $\mathrm{GL}_4$  over  $\mathbb{Q}$ , characterized by the following property : for any prime  $p$  such that  $\pi_p$  is unramified, then  $(\pi^{\mathrm{GL}})_p$  is unramified as well and its Satake parameter is the image of the one of  $\pi_p$  under the map  $r$ . The infinitesimal character of  $(\pi^{\mathrm{GL}})_{\infty}$  is the image of  $\mathrm{inf} \pi_{\infty}$  under the derivative of  $r$ , namely  $\mathfrak{sp}_4(\mathbb{C}) \rightarrow \mathfrak{gl}_4(\mathbb{C})$ . Moreover, there is a unique collection of distinct triplets  $(d_i, n_i, \pi_i)_{i \in I}$ , with integers  $d_i, n_i \geq 1$  and cuspidal selfdual automorphic representations  $\pi_i$  of  $\mathrm{GL}_{n_i}$  with

$$\pi^{\mathrm{GL}} \simeq \boxplus_{i \in I} (\boxplus_{l=0}^{d_i-1} \pi_i \otimes |\cdot|^{\frac{d_i-1}{2}-l}) \quad \text{and} \quad 4 = \sum_{i \in I} n_i d_i.$$

The selfdual representation  $\pi_i$  is symplectic in Arthur’s sense if, and only if,  $d_i$  is odd. All of this is included in [1, Thm. 1.5.2].

(c) For  $\pi$  as in (b), then  $\mathrm{inf} \pi_{\infty}$  is the union, over all  $i \in I$  and all  $0 \leq l < d_i$ , of the multisets  $\frac{d_i-1}{2} - l + \mathrm{inf} (\pi_i)_{\infty}$ . In particular, if we have  $\lambda \in \frac{1}{2}\mathbb{Z}$  and  $\lambda - \mu \in \mathbb{Z}$  for all  $\lambda, \mu \in \mathrm{inf} \pi_{\infty}$ , then  $\mathrm{inf} (\pi_i)_{\infty}$  has the same property for each  $i$  : such a  $\pi_i$  is called *algebraic*. If  $\omega$  is a cuspidal selfdual algebraic automorphic representation of  $\mathrm{GL}_m$  over  $\mathbb{Q}$ , then  $\omega_{\infty}$  is tempered by the Jacquet-Shalika estimates, and its Langlands parameter is trivial on the central subgroup  $\mathbb{R}_{>0}$  of  $\mathrm{W}_{\mathbb{R}}$  (this is the so-called *Clozel purity lemma*, see e.g. [6, Chap. VIII Prop. 2.13]).

(d) The only selfdual cuspidal automorphic representation  $\pi$  of  $\mathrm{GL}_1$  over  $\mathbb{Q}$  such that  $\pi_p$  is unramified for each prime  $p$  is the trivial Hecke character 1 (which is of course selfdual orthogonal). Moreover, for any integer  $k \geq 1$ , the number of cuspidal automorphic representations  $\pi$  of  $\mathrm{GL}_2$  over  $\mathbb{Q}$  such that  $\pi_p$  is unramified for each prime  $p$ , and with  $\mathrm{inf} \pi_{\infty} = \{ -\frac{k-1}{2}, \frac{k-1}{2} \}$ , is the dimension

of the space  $S_k(\Gamma_1)$  of cuspidal modular forms of weight  $k$  for  $\Gamma_1 = \text{SL}_2(\mathbb{Z})$ . Indeed, this is well-known for  $k > 1$ , and for  $k = 1$  it follows from the fact that there is no Maass form of eigenvalue  $1/4$  for  $\text{SL}_2(\mathbb{Z})$  (a fact due to Selberg, see also [6, Chap. IX §3.19] for a short proof).

*Proof.* (of the vanishing of  $S_{j,1}(\Gamma_2)$  for any  $j$ ). It is enough to show that there is no discrete automorphic representation  $\pi$  of  $\text{PGSp}_4$  over  $\mathbb{Q}$  which is unramified at every prime and with  $\pi_\infty \simeq U_{j,1}$ . For that we study  $\pi^{\text{GL}}$ , and the associated collection  $(d_i, n_i, \pi_i)_{i \in I}$  given by (b) above. By (a), the infinitesimal character of  $(\pi^{\text{GL}})_\infty$  is

$$\inf U_{j,1} = \left\{ \frac{j+1}{2}, \frac{j-1}{2}, -\frac{j-1}{2}, -\frac{j+1}{2} \right\}.$$

If we have  $d_i = 1$  for each  $i$ , then  $(\pi^{\text{GL}})_\infty$  is tempered by (c), hence so is  $\pi_\infty$  by Arthur’s local-global compatibility [1, Thm. 1.5.1 (b) & Thm. 1.5.2], a contradiction as  $U_{j,1}$  is non-tempered by (a). Fix  $i \in I$  with  $d_i \geq 2$ . If we have  $n_i \geq 2$  then we must have  $I = \{i\}$  and  $n_i = d_i = 2$  by the equality  $4 = \sum_i n_i d_i$ . By (c) and the shape of  $\inf U_{j,1}$  above, we necessarily have  $\inf(\pi_i)_\infty = \{j/2, -j/2\}$ . But this is absurd by (d) and the vanishing  $S_{j+1}(\Gamma_1) = 0$  for any even integer  $j \geq 0$ . We have thus  $(d_i, n_i, \pi_i) = (2, 1, 1)$ . Choose  $i' \in I - \{i\}$ . As we have  $(d_{i'}, n_{i'}, \pi_{i'}) \neq (d_i, n_i, \pi_i)$ , the previous argument shows  $d_{i'} = 1$ , so  $\pi_{i'}$  is symplectic by (b), which forces  $n_{i'}$  to be even, hence the only possibility is  $n_{i'} = 2$  and  $I = \{i, i'\}$ . But then the shape of  $\inf U_{j,1}$  and (c) show that we have either  $j = 0$  and  $\inf(\pi_{i'})_\infty = \{1/2, -1/2\}$  or  $j = 1$  and  $\inf(\pi_{i'})_\infty = \{3/2, -3/2\}$ . Both cases are absurd by (d) as we have  $S_4(\Gamma_1) = S_2(\Gamma_1) = 0$ , and we are done.  $\square$

The second assertion of the proposition will be a consequence of the following two lemmas.

LEMMA A.2. *Let  $j \geq 0$  be an even integer. The integer  $\dim S_{j,2}(\Gamma_2)$  is the number of cuspidal, selfdual symplectic, automorphic representations  $\Pi$  of  $\text{GL}_4$  over  $\mathbb{Q}$  whose local components  $\Pi_p$  are unramified for each prime  $p$ , and with  $\inf \Pi_\infty = \left\{ \frac{j+1}{2}, \frac{j+1}{2}, -\frac{j+1}{2}, -\frac{j+1}{2} \right\}$ .*

*Proof.* Let  $\pi$  be a cuspidal automorphic representation of  $\text{PGSp}_4$  over  $\mathbb{Q}$  generated by an arbitrary Hecke eigenform  $F$  in  $S_{j,2}(\Gamma_2)$ . Consider its associated automorphic representation  $\pi^{\text{GL}}$  of  $\text{GL}_4$  and collection of  $(d_i, n_i, \pi_i)$ ’s as in (b). We claim that  $\pi^{\text{GL}}$  is necessarily cuspidal, i.e.  $I = \{i\}$  is a singleton and  $d_i = 1$ , so that  $\Pi = \pi^{\text{GL}}$  satisfies all the assumptions of the statement by (a) and (b).

Let us show first that we have  $d_i = 1$  for each  $i \in I$ . Otherwise, (c) shows that two elements of  $\inf U_{j,2}$  must differ by 1, which only happens for  $j = 0$ . But for  $j = 0$  an argument similar to the one in the previous proof shows that if  $d_i > 1$  then we have  $I = \{i, i'\}$  with  $(d_i, n_i, \pi_i) = (2, 1, 1)$ ,  $n_{i'} = 2$  and  $\inf(\pi_{i'})_\infty = \{1/2, -1/2\}$ , which is absurd by the vanishing  $S_2(\Gamma_1) = 0$  and (d), and we are done. As a consequence, the Langlands parameter of  $(\pi^{\text{GL}})_\infty$  is  $I_{j+1} \oplus I_{j+1}$  by (c).

Let us denote by  $\psi$  Arthur's substitute for the *global parameter* of the representation  $\pi$  of  $\mathrm{PGSp}_4$  defined in [1, Chap. 1 §1.4]. We have just proved that  $\psi_\infty$  is the tempered Langlands parameter of  $\mathrm{PGSp}_4(\mathbb{R})$  with  $r \circ \psi_\infty \simeq \mathbf{I}_{j+1} \oplus \mathbf{I}_{j+1}$ , *i.e.* the Langlands parameter of  $U_{j,2}$  by (a). We have  $\mathcal{S}_{\psi_\infty} = \mathbb{Z}/2\mathbb{Z}$ : the corresponding  $L$ -packet of  $\mathrm{PGSp}_4(\mathbb{R})$  (limit of discrete series) has two elements, namely  $U_{j,2}$  and a generic limit of discrete series with same infinitesimal character. We now apply Arthur's multiplicity formula to the element  $\pi$  of the global packet  $\Pi_\psi$  defined by Arthur. As we have  $d_i = 1$  for all  $i$ , either  $\pi^{\mathrm{GL}}$  is cuspidal or we have  $I = \{i, i'\}$  with  $n_i = n_{i'} = 2$  and  $\inf(\pi_i)_\infty = \inf(\pi_{i'})_\infty = \{\frac{j+1}{2}, -\frac{j+1}{2}\}$ . If  $\pi^{\mathrm{GL}}$  is not cuspidal, then according to Arthur's definitions the natural map  $\mathcal{S}_\psi \rightarrow \mathcal{S}_{\psi_\infty}$  is an isomorphism of groups of order 2. But then his multiplicity formula shows that  $\pi_\infty$  has to be generic since  $\pi_p$  is unramified for each prime  $p$ , a contradiction as  $U_{j,2}$  is not generic. (We have shown that  $\pi$  is not of "Yoshida type".) We have thus proved that  $\pi^{\mathrm{GL}}$  is cuspidal. Note that in this case we have  $\mathcal{S}_\psi = 1$ , thus by the multiplicity formula again, the multiplicity of  $\pi$  in the automorphic discrete spectrum of  $\mathrm{PGSp}_4$  is 1; in particular, the Hecke eigenspace of the eigenform  $F$  we started from has dimension 1. It thus only remains to show that any  $\Pi$  as in the statement is in the image of the construction of the first paragraph above.

Let  $\Pi$  be as in the statement. The Langlands parameter of  $\Pi_\infty$  is the image under  $r$  of the one of  $U_{j,2}$  by (a) and (c). A trivial application of Arthur's multiplicity formula shows the existence of a discrete automorphic  $\pi$  for  $\mathrm{PGSp}_4$  with  $\pi_\infty \simeq U_{j,2}$ , which is unramified at every prime, and satisfying  $\pi^{\mathrm{GL}} \simeq \Pi$ . As  $U_{j,2}$  is tempered, a classical result of Wallach ensures that  $\pi$  is actually cuspidal, hence generated by an element of  $S_{j,2}(\Gamma_2)$ : this concludes the proof.  $\square$

LEMMA A.3. *For any even integer  $0 \leq j \leq 38$  there is no  $\Pi$  as in Lemma A.2.*

In order to contradict the existence of a  $\Pi$  as in Lemma A.2 for small  $j$ , and following work of Odlyzko, Mestre, Fermigier, Miller and Chenevier-Lannes, we shall apply the so-called *explicit formula* "à la Riemann-Weil" to a suitable test function  $F$  and to the complete Rankin-Selberg  $L$ -function  $L(s, \Pi \times \Pi')$ , first to  $\Pi' = \Pi^\vee$  (the contragredient of  $\Pi$ ) and then to some other well-chosen cuspidal automorphic representations  $\Pi'$ . Let us stress that the analytic properties of those Rankin-Selberg  $L$ -functions (meromorphic continuation to  $\mathbb{C}$ , functional equation, determination of the poles, and boundedness in vertical strips away from the poles) which have been established by Gelbart, Jacquet, Shalika and Shahidi, will play a crucial role in the argument.

*Proof.* It will be convenient to follow the exposition of the explicit formula given in [6, Chap. IX §3], which is designed for this kind of applications, and from which we shall borrow our notations. In particular, we choose for the test function  $F$  the scaling of Odlyzko's function which is denoted by  $F_\lambda$  in [6, Chap. IX §3.16], and we denote by  $K_\infty$  the Grothendieck ring of finite dimensional complex representations of the quotient of the compact group  $W_{\mathbb{R}}$

by its central subgroup  $\mathbb{R}_{>0}$ , and by  $J_F : K_\infty \rightarrow \mathbb{R}$  the concrete linear form associated to  $F$  defined in Proposition-Def. 3.7 *loc. cit.*

Let  $j \geq 0$  be an even integer and let  $\Pi$  be as in Lemma A.2. As already explained, it follows from (c) that the Langlands parameter of  $\Pi_\infty$  is  $I_{j+1} \oplus I_{j+1}$ , whose square in the ring  $K_\infty$  is  $4(I_{2j+2} + I_0)$ . The explicit formula applied to  $\Pi \times \Pi^\vee$  leads to the inequality [6, Chap. IX Cor. 3.11 (i)] :

$$J_{F_\lambda}(I_{2j+2}) + J_{F_\lambda}(I_0) \leq \frac{2}{\pi^2} \lambda$$

for all  $\lambda > 0$ . As  $J_{F_\lambda}(I_w)$  is a non-increasing function of  $w$ , the truth of the proposition for  $j \leq 34$  is a consequence of the following numerical computation for  $\lambda = 3.3$

$$J_{F_\lambda}(I_{70}) + J_{F_\lambda}(I_0) \simeq 0.679 \quad \text{and} \quad \frac{2}{\pi^2} \lambda \simeq 0.669$$

(the values are given up to  $10^{-3}$ , and the left-hand side has been computed using the formula for  $J_{F_\lambda}$  given in [6, Chap. IX Prop. 3.17]).

We now explain how to deal with the cases  $j = 36$  and  $38$ . By Tsushima's formula (proved for  $k = 3$  independently by Petersen and Taïbi), we know that the first value of  $j$  such that  $S_{j,3}(\Gamma_2)$  is non-zero is  $j = 36$ , in which case it has dimension 1. Let  $\pi$  be the cuspidal automorphic representation of  $\text{PGSp}_4$  over  $\mathbb{Q}$  generated by  $S_{36,3}(\Gamma_2)$  and set  $\Pi' = \pi^{\text{GL}}$ . This  $\Pi'$  is a selfdual cuspidal representation by [6, Chap. IX Prop. 1.4], and the Langlands parameter of  $\Pi'_\infty$  is  $I_{39} \oplus I_{37}$  (the existence of such a  $\Pi'$  actually "explains" why the argument above breaks down at  $j = 36$ , as the Langlands parameter of  $\Pi'_\infty$  is "close" to  $I_{37} \oplus I_{37}$ ). We now apply the explicit formula to  $\Pi \times \Pi^\vee$ ,  $\Pi' \times \Pi'^\vee$  and  $\Pi \times \Pi'^\vee$ . It leads to a simple criterion, given in [6, Chap. IX Scholie 3.26], for  $\Pi$  not to exist : the explicit quantity denoted there by  $t(V, V', \lambda)$  has to be  $\geq 0$  for all  $\lambda$ , where  $V$  and  $V'$  are the respective Langlands parameter of  $\Pi_\infty$  and  $\Pi'_\infty$ . But a computation gives

$$t(I_{37} + I_{37}, I_{39} + I_{37}, 4) \simeq -0.429 \quad \text{and} \quad t(I_{39} + I_{39}, I_{39} + I_{37}, 4) \simeq -0.039$$

(these values are given up to  $10^{-3}$ ) which are both  $< 0$ . This concludes the proof.  $\square$

*Remark A.4.* (i) As we have  $S_{j,3}(\Gamma_2) = 0$  for  $j < 36$  by Tsushima's formula, the vanishing of  $S_{j,2}(\Gamma_2)$  for  $j \leq 38$  is a very mild evidence toward Conjecture 1.1 of Cléry and van der Geer.

(ii) Lemma A.2 and the explicit formula can also be used to obtain upper bounds on  $\dim S_{j,2}(\Gamma_2)$ . Indeed, keeping the notations in the above proof, and applying [6, Chap. IX Cor. 3.14] (due to Taïbi), we get that the inequality

$$(J_{F_\lambda}(I_{2j+2}) + J_{F_\lambda}(I_0)) \dim S_{j,2}(\Gamma_2) \leq \frac{2}{\pi^2} \lambda$$

holds for all  $\lambda > 0$ . When the parenthesis on left hand side is  $> 0$ , which happens (for big enough  $\lambda$ ) for all  $j \leq 138$ , we obtain an explicit upper bound for  $\dim S_{j,2}(\Gamma_2)$ . For instance, we get  $\dim S_{j,2}(\Gamma_2) \leq 1$  for all  $j < 54$  and  $\dim S_{j,2}(\Gamma_2) \leq 2$  for all  $j < 66$  (choose respectively  $\lambda = 5$  and  $\lambda = 6$ ).

Our last and main result concerns the kernel  $\Gamma_2[2]$  of the reduction  $\mathrm{Sp}_4(\mathbb{Z}) \rightarrow \mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z})$ .

**THEOREM A.5.** *We have  $S_{j,1}(\Gamma_2[2]) = 0$  for any  $j$ .*

Our proof will be an elaboration of the one of Proposition A.1. We shall also use the vanishing  $S_{j,1}(\Gamma_2[2]) = 0$  for  $j \leq 8$ , proved by Cléry and van der Geer in this paper (Proposition 7.1).

*Proof.* As we have  $-1 \in \Gamma_2[2]$  we may also assume  $j$  is even. Let us denote by  $J$  the principal congruence subgroup of  $\mathrm{PGSp}_4(\mathbb{Z}_2)$  and by  $\mathbb{A}$  the adèle ring of  $\mathbb{Q}$ ; we easily check  $\mathrm{PGSp}_4(\mathbb{A}) = \mathrm{PGSp}_4(\mathbb{Q}) \cdot (\mathrm{PGSp}_4(\mathbb{R})^0 \times J \times \prod_{p \neq 2} \mathrm{PGSp}_4(\mathbb{Z}_p))$ . Moreover, classical arguments show that we have  $S_{j,1}(\Gamma_2[2]) = 0$  if, and only if, there is no cuspidal automorphic representation  $\pi$  of  $\mathrm{PGSp}_4$  over  $\mathbb{Q}$  which is unramified at every odd prime, such that  $\pi_2$  has a non-zero invariant under  $J$ , and with  $\pi_\infty \simeq U_{j,1}$ . Thus we fix such a  $\pi$  and consider  $\pi^{\mathrm{GL}}$ , as well as the associated collection  $(d_i, n_i, \pi_i)_{i \in I}$ , given by (b) above.

By the same argument as in the case  $\Gamma = \Gamma_2$ , there exists  $i \in I$  with  $d_i \geq 2$ . If we have  $n_i = 1$ , which forces  $\inf \pi_i = \{0\}$ , we must have  $d_i = 2$  and  $j \in \{0, 2\}$  by the shape of  $\inf U_{j,1}$ . But this is a contradiction as  $S_{j,1}(\Gamma_2[2]) = 0$  for  $j = 0, 2$ . So we must have  $n_i = d_i = 2$ ,  $I = \{i\}$ , and  $\pi_i$  is orthogonal with  $\inf(\pi_i)_\infty = \{-j/2, j/2\}$ .

The classification of orthogonal cuspidal automorphic representations of  $\mathrm{GL}_2$  over  $\mathbb{Q}$ , a very special case of Arthur’s results, is well-known. First of all, the central character of such a representation has order 2, hence corresponds to some uniquely defined quadratic extension  $K$  of  $\mathbb{Q}$ . Moreover, for any Hecke character  $\chi$  of  $K$  which is trivial on the idele group of  $\mathbb{Q}$ , and with  $\chi^2 \neq 1$ , the automorphic induction of  $\chi$  to  $\mathbb{Q}$  is an orthogonal cuspidal automorphic representation of  $\mathrm{GL}_2$  over  $\mathbb{Q}$  that we shall denote by  $\mathrm{ind}(\chi)$ . It turns out that they all have this form, and that we have moreover  $\mathrm{ind}(\chi) \simeq \mathrm{ind}(\chi')$  if, and only if, we have  $\chi = \chi'$  or  $\chi^{-1} = \chi'$ . Last but not least, to any  $\chi$  as above Arthur associates a global packet  $\Pi(\chi) = \bigotimes'_v \Pi_v(\chi_v)$  of irreducible admissible representations of  $\mathrm{PGSp}_4$  over  $\mathbb{Q}$ , whose elements are exactly the discrete automorphic representations  $\omega$  which satisfy  $\omega^{\mathrm{GL}} = \mathrm{ind}(\chi) \otimes |\cdot|^{1/2} \boxplus \mathrm{ind}(\chi) \otimes |\cdot|^{-1/2}$  (*Soudry type*); in this “stable case” any element of  $\Pi(\chi)$  is automorphic by Arthur’s multiplicity formula.

Going back to our specific situation, let  $K$  and  $\chi$  be such that  $\pi_i \simeq \mathrm{ind}(\chi)$ . Since  $\pi_i$  is unramified outside 2, then so is  $K$  and we necessarily have

$$K = \mathbb{Q}(\sqrt{d}) \text{ with } d \in \{-2, -1, 2\}.$$

We first claim  $d \neq 2$ . Indeed, a Hecke character of real quadratic field has the form  $|\cdot|^{s_0} \chi_0$  with  $\chi_0$  a finite order character and  $s_0 \in \mathbb{C}$ . We would thus have  $\{s_0, s_0\} = \inf(\pi_i)_\infty = \{j/2, -j/2\}$ , which implies  $s_0 = j = 0$ , which is again absurd. So  $K$  is imaginary quadratic. As  $\chi_\infty$  is trivial on  $\mathbb{R}^\times$  by assumption on  $\chi$ , and up to replacing  $\chi$  by  $\chi^{-1}$  if necessary, the shape of  $\inf(\pi_i)_\infty$  implies then  $\chi_\infty(z) = (z/\bar{z})^{j/2}$ .

Here comes the main trick. Let  $\eta$  be the Hecke character of the statement of Lemma A.6 below, and set  $w_K = |\mathcal{O}_K^\times|$ . We may assume  $j \geq 2$ , so there are unique integers  $r$  and  $j'/2$ , with  $r \geq 0$  and  $1 \leq j'/2 \leq w_K$ , such that  $j/2 = r w_K + j'/2$ . Consider the Hecke character  $\chi' = \chi \eta^{-r}$  of  $K$ . It is obviously trivial on the idele group of  $\mathbb{Q}$ , and it satisfies  $(\chi')^2 \neq 1$  as we have  $\chi'_\infty(z) = (z/\bar{z})^{j'/2}$  with  $j' > 0$ . Consider now the packet  $\Pi(\chi')$ . As we have  $\chi_2 = \chi'_2$ , the local Arthur packets  $\Pi_2(\chi_2)$  and  $\Pi_2(\chi'_2)$  do coincide. As  $\pi$  belongs to  $\Pi(\chi)$ , its local component  $\pi_2$  also belongs to  $\Pi_2(\chi_2)$ . We may thus consider a representation  $\pi' = \bigotimes'_v \pi'_v$  in  $\Pi(\chi')$  with  $\pi'_2 \simeq \pi_2$ , with  $\pi'_p$  unramified for all odd prime  $p$  (since  $\chi'_p$  is unramified for such a  $p$ ), and with  $\pi'_\infty \simeq U_{j',1}$ . This last property holds because the Langlands packet associated to the Arthur packet  $\Pi_\infty(\chi'_\infty)$ , which is included in  $\Pi_\infty(\chi'_\infty)$  by [1, Prop. 7.4.1], is the one of  $U_{j',1}$  by Remark (a) above. As already explained, the representation  $\pi'$  is discrete automorphic by Arthur, and even cuspidal as its Archimedean component is tempered. As  $\pi'_2 \simeq \pi_2$  has non-zero invariants under the principal congruence subgroup  $J$  of  $\mathrm{PGSp}_4(\mathbb{Z}_2)$ , it follows that  $\pi'$  is generated by an element in  $S_{j',1}(\Gamma_2[2])$  by the first paragraph above. But now we have the inequality  $j' \leq 2w_K \leq 8$ , a contradiction by the vanishing  $S_{j',1}(\Gamma_2[2]) = 0$  for  $j' \leq 8$ .  $\square$

LEMMA A.6. *Let  $K = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$  with  $d = -1, -2$  and set  $w_K = |\mathcal{O}_K^\times|$ . There is a Hecke character  $\eta$  of  $K$  which is unramified outside  $\{2, \infty\}$  and which satisfies  $\eta_2 = 1$  and  $\eta_\infty(z) = (z/\bar{z})^{w_K}$  for all  $z \in K_\infty^\times$ . Moreover,  $\eta$  is trivial on the idele group of  $\mathbb{Q}$ .*

*Proof.* Denote by  $\mathbb{A}_F$  the adèle ring of the number field  $F$ . As  $\mathcal{O}_K$  has class number 1 we have the decomposition  $\mathbb{A}_K^\times = K^\times \cdot (K_\infty^\times \times K_2^\times \times \prod_{v \neq 2, \infty} \mathcal{O}_{K_v}^\times)$ . This implies first the (unrequired) uniqueness of  $\eta$ , and shows that its existence is equivalent to the fact that the morphism  $z \mapsto (z/\bar{z})^{w_K}, \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ , is trivial on the subgroup  $(\mathcal{O}_K[1/2])^\times$ . This is indeed the case as this latter group is generated by  $\mathcal{O}_K^\times$  and by some element  $\pi \in \mathcal{O}_K$  with norm 2 and which satisfies  $\pi/\bar{\pi} \in \mathcal{O}_K^\times$ . The last assertion follows from the equality  $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot (\mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times)$  and the properties of  $\eta$ .  $\square$

#### APPENDIX B. PROOF OF THEOREM 1.1 BY GAËTAN CHENEVIER

In this second appendix, we explain how to deduce Theorem 1.1 stated in the paper of Cléry and van der Geer from the works of Rösner [20] and Weissauer [26], for the convenience of the reader.

We first recall some results on Yoshida lifts taken from Weissauer's work [26, Chap. 4 & 5]. Let us fix  $f$  and  $g$  two non-proportional elliptic eigen newforms of same even weight  $j+2$ , and let us denote by  $\pi$  and  $\pi'$  the (distinct) cuspidal automorphic representations of  $\mathrm{GL}_2(\mathbb{A})$  that they generate,  $\mathbb{A}$  being the adèle ring of  $\mathbb{Q}$ . In particular,  $\pi_\infty$  and  $\pi'_\infty$  are isomorphic discrete series, and we may assume that  $\pi$  and  $\pi'$  are normalized such that this discrete series has trivial central character (as  $j$  is even). For Yoshida lifts  $Y(f, g)$  of  $f$  and  $g$  to exist,



we also need to assume that  $f$  and  $g$  have the same *nebensystem*, i.e. that  $\pi$  and  $\pi'$  have the same central character.

Let  $\Pi(\pi, \pi') = \bigotimes'_v \Pi_v(\pi_v, \pi'_v)$  be the restricted tensor product, over all the places  $v$  of  $\mathbb{Q}$ , of the local  $L$ -packet  $\Pi_v(\pi_v, \pi'_v)$  of irreducible admissible representations of  $\mathrm{GSp}_4(\mathbb{Q}_v)$  associated to the pair  $\{\pi_v, \pi'_v\}$  by Weissauer. For each place  $v$  of  $\mathbb{Q}$ , this local  $L$ -packet has either 1 or 2 elements, including a unique *generic* element; it has another element if, and only if, both  $\pi_v$  and  $\pi'_v$  are discrete series [26, §4.10.3] [20, Lemma 4.5]. In particular, the (Langlands) Archimedean packet  $\Pi_\infty(\pi_\infty, \pi'_\infty)$  has two elements, the non-generic one being the holomorphic limit of discrete series  $U_{j,2}$  recalled in appendix A. Also, if both  $\pi$  and  $\pi'$  are unramified at the finite place  $v$  then  $\Pi_v(\pi_v, \pi'_v)$  is a singleton (thus  $\Pi(\pi, \pi')$  is finite). The multiplicity formula proved by Weissauer [26, Thm. 5.2, p. 186] states that an element  $\varpi$  of  $\Pi(\pi, \pi')$  is discrete automorphic if, and only if, there is an even number of places  $v$  such that  $\varpi_v$  is non-generic. He also shows that such an element has multiplicity one in the discrete spectrum of  $\mathrm{GSp}_4$ ; it is necessarily cuspidal as the two elements of  $\Pi_\infty(\pi_\infty, \pi'_\infty)$  are tempered.

Let us denote by  $J \subset \mathrm{GSp}_4(\mathbb{Z}_2)$  the principal congruence subgroup. Some classical arguments show that the Yoshida lifts  $Y(f, g)$  which belong to the space  $YS_{j,2}^{s[w]}$  of the statement are in natural bijection with certain vectors of the finite part of the cuspidal automorphic representations  $\varpi$  in  $\Pi(\pi, \pi')$  having the following properties :

- (i)  $\varpi_\infty \simeq U_{j,2}$ ,
- (ii)  $\varpi_p^{\mathrm{GSp}_4(\mathbb{Z}_p)} \neq 0$  for  $p > 2$  (in which case we have  $\dim \varpi_p^{\mathrm{GSp}_4(\mathbb{Z}_p)} = 1$ ),
- (iii) the  $s[w]$ -isotypic component  $\varpi_2^J$  is non-zero.

More precisely, the Yoshida lifts  $Y(f, g)$  corresponding to such a  $\varpi$  form a linear subspace  $YS_{j,2}^{s[w]}[\varpi] \subset YS_{j,2}^{s[w]}$  isomorphic to the  $s[w]$ -isotypic component of  $\varpi_2^J$  as an  $S_6$ -representation. The space  $YS_{j,2}^{s[w]}$  of the statement is then the direct sum of its subspaces  $YS_{j,2}^{s[w]}[\varpi]$  where  $f, g$  and  $\varpi$  vary, with  $\varpi$  cuspidal automorphic satisfying (i), (ii) and (iii).

We still fix elliptic newforms  $f$  and  $g$  as above, hence  $\pi$  and  $\pi'$  as well. By [20, Cor. 4.14], we know first that  $\Pi_p(\pi_p, \pi'_p)$  has an element with non-zero invariants under  $\mathrm{GSp}_4(\mathbb{Z}_p)$  if, and only if, both  $\pi_p$  and  $\pi'_p$  are unramified. Moreover, the same corollary asserts that if  $\Pi_2(\pi_2, \pi'_2)$  has an element with non-zero  $J$ -invariants, then both  $\pi_2$  and  $\pi'_2$  have non-zero invariants under the principal congruence subgroup of  $\mathrm{GL}_2(\mathbb{Z}_2)$ . As a first consequence, if  $\varpi \in \Pi(\pi, \pi')$  does satisfy (ii) and (iii) then  $f$  and  $g$  are newforms on  $\Gamma_0(N)$  with  $N|4$  (and both  $\pi$  and  $\pi'$  have a trivial central character). Moreover, by the statement recalled above concerning the multiplicity formula ("even parity of the number of non-generic places"), there is at most one cuspidal automorphic  $\varpi \in \Pi(\pi, \pi')$  satisfying (i), (ii) and (iii) above, and it has the property that  $\varpi_2$  is the non-generic element of  $\Pi_2(\pi_2, \pi'_2)$ . In particular, this latter  $L$ -packet has

two elements and both  $\pi_2$  and  $\pi'_2$  are discrete series of  $\mathrm{GL}_2(\mathbb{Q}_2)$  (thus neither  $f$  nor  $g$  can have level 1).

By Rösner [20, Lemma 5.22], there are only three possibilities for the isomorphism class of a representation of  $\mathrm{PGL}_2(\mathbb{Q}_2)$  generated by an elliptic newform of level  $\Gamma_0(2)$  or  $\Gamma_0(4)$ , namely the *Steinberg* representation  $\mathrm{St}$  and its unramified quadratic twist  $\mathrm{St}'$  in level  $\Gamma_0(2)$ , and a certain supercuspidal representation  $\mathrm{Sc}$  in level  $\Gamma_0(4)$ . In particular, they are all discrete series. Assume now that  $\sigma$  and  $\sigma'$  are (possibly equal) elements of the set  $\{\mathrm{St}, \mathrm{St}', \mathrm{Sc}\}$  and let  $\tau$  be the non-generic element of  $\Pi_2(\sigma, \sigma')$ . View the finite dimensional vector space  $\tau^J$  as a representation of  $S_6$ . In order to prove the theorem it only remains to show that either  $\tau^J$  is 0 or we are in exactly one of the following situations :

- (a)  $\{\sigma, \sigma'\} = \{\mathrm{St}, \mathrm{St}'\}$  and  $\tau^J \simeq s[1^6]$ ,
- (b)  $\{\sigma, \sigma'\} = \{\mathrm{Sc}\}$  and  $\tau^J \simeq s[2, 1^4]$ ,
- (c)  $\{\sigma, \sigma'\} = \{\mathrm{St}\}$  or  $\{\mathrm{St}'\}$  and  $\tau^J \simeq s[2^3]$ .

This is a delicate analysis which fortunately has been carried out by Rösner. Indeed, this is exactly the content of the left-bottom part of [20, Table 4.2, p. 63] with  $q = 2$ . This table shows that there are just three non-zero possible representations for  $\tau^J$ , denoted by  $\theta_2$ ,  $\theta_5$  and  $\chi_9(1)$  there but which correspond respectively to the representations  $s[2^3]$ ,  $s[1^6]$  and  $s[2, 1^4]$  by Rösner's other table [20, Table 5.2, p. 103], exactly according to the three cases above (read the table with  $\Pi_1 = \mathrm{Sc}$ ,  $\xi_\mu \mathrm{St} = \mathrm{St}'$ , and take for  $\mu$  either the trivial character of  $\mathbb{Q}_2^\times$  or its unramified quadratic character).  $\square$

ACKNOWLEDGEMENT. Gaëtan Chenevier is supported by the C.N.R.S. and by the project ANR-14-CE25 (PerCoLaTor). He would like to thank Fabien Cléry and Gerard van der Geer for inviting him to include these two appendices to their paper.

LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UNIVERSITÉ PARIS-SUD, UNIVERSITÉ PARIS-SACLAY, 91405 ORSAY, FRANCE.

*E-mail address:* `gaetan.chenevier@math.cnrs.fr`

#### REFERENCES

- [1] J. Arthur: *The endoscopic classification of representations: orthogonal and symplectic groups*, Colloquium Publ. 61, Amer. Math. Soc. (2013).
- [2] J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of degree two*. 2016, URL= <http://smf.compositio.nl>
- [3] J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of genus 2 and level 2: conjectures and cohomological computations*. IMRN, Art. ID rnn 100 (2008).
- [4] J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of degree three and the cohomology of local systems*. *Selecta Math.* 20 (2014), 83-124.
- [5] S. Boecherer, R. Schulze-Pillot: *Siegel modular forms and theta series attached to quaternion algebras. II*. *Nagoya Math. J.* 147 (1997), 71–106.
- [6] G. Chenevier, J. Lannes: *Automorphic forms and even unimodular lattices*. To appear in: *Ergebnisse der Mathematik und ihrer Grenzgebiete*.

- [7] F. Cléry, C. Faber, G. van der Geer: *Covariants of binary sextics and vector-valued Siegel modular forms of genus two*. Math. Annalen (2017), 1649–1669; DOI 10.1007/s00208--016--1510--2
- [8] F. Cléry, C. Faber, G. van der Geer: *Covariants of binary sextics and modular forms of degree 2 with a character*. arXiv:1803.05624
- [9] F. Cléry, G. van der Geer: *Constructing vector-valued Siegel modular forms from scalar-valued Siegel modular forms*. Pure and Applied Math. Quarterly 11 (2015), 21–47.
- [10] F. Cléry, G. van der Geer, S. Grushevsky: *Siegel modular forms of genus 2 and level 2*. International Journal of Mathematics 26, No. 5 (2015) 1550034 (51 pages)
- [11] M. Eichler, D. Zagier: *The theory of Jacobi forms*, 1985, Birkhäuser.
- [12] C. Faber, G. van der Geer: *Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes*, I, II. C.R. Acad. Sci. Paris, Sér. I 338 (2004), 381–384, 467–470.
- [13] G. Faltings, C-L. Chai: *Degeneration of abelian varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22. Springer-Verlag, Berlin, 1990.
- [14] G. van der Geer: *Siegel modular forms and their applications*. In: J. Bruinier, G. van der Geer, G. Harder, D. Zagier: *The 1-2-3 of modular forms*. Springer Verlag, 2008.
- [15] J. H. Grace, A. Young: *The algebra of invariants*. 1903, Cambridge: Cambridge University Press
- [16] T. Ibukiyama: *Vector-valued Siegel modular forms of symmetric tensor weight of small degrees*, Commentarii Math. Univ. Sancti Pauli 61 (2012).
- [17] T. Ibukiyama, S. Wakatsuki: *Siegel modular forms of small weight and the Witt operator*. Contemp. Math. 493 (2009), 189–209.
- [18] J.-I. Igusa: *Modular forms and projective invariants*. Amer. J. Math. 89, 817–855 (1967)
- [19] N. Katz: *p-adic properties of modular schemes and modular forms*. In: *Modular functions of one variable, III*. Lecture Notes in Mathematics, 350, Berlin, New York: Springer-Verlag, pp. 69–190.
- [20] M. Rösner: *Parahoric restriction for GSp(4) and the inner cohomology of Siegel modular threefolds*. Inaugural dissertation, Ruprecht-Karls-Universität Heidelberg, 2016.
- [21] A. Saha, R. Schmidt: *Yoshida lifts and simultaneous non-vanishing of dihedral twists of modular L-functions*. J. Lond. Math. Soc. (2)88 (2013), 251–270.
- [22] R. Schmidt: *Archimedean aspects of Siegel modular forms of degree 2*. Rocky Mountain J. Math. (2017).
- [23] N.-P. Skoruppa: *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts*. Inaugural-Dissertation, Bonner Mathematische Schriften 159, Bonn, 1984.
- [24] R. Tsushima: *An explicit dimension formula for the spaces of generalized automorphic forms with respect to Sp(2, Z)*. Proc. Japan Acad. Ser. A Math. Sci. 59 (1983), no. 4, 139–142.
- [25] R. Uchida: *The dimension of the space of Siegel modular forms of weight  $\det^2 \text{Sym}(20)$* . (In Japanese). Master Thesis at Osaka University, 2007 March, pp. 340.
- [26] R. Weissauer: *Endoscopy for GSp(4) and the Cohomology of Siegel Modular Threefolds*. Lecture Notes in Mathematics 1968. Springer, 1st edition (2009).
- [27] H. Yoshida: *On a explicit construction of Siegel modular forms of genus 2*. Proc. Japan Acad. 55 (1979), 297–300.
- [28] H. Yoshida: *Siegel’s modular forms and the arithmetic of quadratic forms*. Invent. math., 60 (1980), 193–248.

Fabien Cléry  
Department of Mathematics  
Loughborough University  
England  
cleryfabien@gmail.com

Gerard van der Geer  
Korteweg-de Vries Instituut  
Universiteit van Amsterdam  
Postbus 94248  
1090 GE Amsterdam  
The Netherlands  
geer@science.uva.nl