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The Cycle Classes of Divisorial Maroni Loci

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We determine the cycle classes of effective divisors in the compactified Hurwitz spaces $\overline{\mathcal{H}}_{d,g}$ of curves of genus g with a linear system of degree d that extend the Maroni divisors on $\mathcal{H}_{d,g}$. Our approach uses Chern classes associated to a global-to-local evaluation map of a vector bundle over a generic $\mathbb{P}^1$-bundle over the Hurwitz space.

1 Introduction

Let C be a smooth projective curve of genus g and let $\gamma : C \to \mathbb{P}^1$ exhibit C as a simply-branched cover of degree d of the projective line. Then we can associate to γ a $(d-1)$-tuple $\alpha = (a_1, \ldots, a_{d-1})$ of integers with $\sum_i a_i = g + d - 1$ by considering the kernel of the trace map $\gamma_* \mathcal{O}_C \to \mathcal{O}_{\mathbb{P}^1}$; this is a locally free sheaf of rank $d-1$ on $\mathbb{P}^1$; hence by Birkhoff–Grothendieck its dual can be written as $\oplus_i \mathcal{O}(a_i)$, where we may assume that the a_i are listed in non-decreasing order. For sufficiently general C and when $d < g$ this $(d-1)$-tuple $\alpha = (a_1, \ldots, a_{d-1})$ has a geometric interpretation as the type of the scroll in the canonical space $\mathbb{P}^{g-1}$ of C with fibres the $\mathbb{P}^{d-2}$ spanned by the fibres of γ. The invariant α allows one to define a stratification on the Hurwitz space $\mathcal{H}_{d,g}$ of genus g degree d simply-branched covers of $\mathbb{P}^1$, with the strata corresponding to these $(d-1)$-tuples α.

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In the case of trigonal covers this stratification was studied by Maroni (see [14]) and the stratification is named after him. The numbers a_i are sometimes called the scrollar invariants of the cover γ and seem to appear first in a paper by Christoffel dating from 1878 (see [3, pp. 242–243]) and alternatively can be characterized as the numbers n where the function $h^1((n-1)D) - h^1(nD)$ jumps with D a divisor of the linear system given by γ. These invariants attracted attention by the work of Schreyer [17]. A result of Ballico [1] implies that for a general cover γ of genus

$$g = (d-1)k + s \qquad \text{with} \quad 0 \leq s < d-1$$

the $(d-1)$-tuple α is of the form $(k+1, k+1, \ldots, k+1, k+2, \ldots, k+2)$ with $d-1-s$ terms equal to $k+1$. If $s = 0$ one expects that there is a divisor where the type α jumps and this holds indeed as was proved by Coppens and Martens in [4]; see also [15] and [7, p. 13/14]. For an overview of the theory of these Maroni loci we refer to the thesis of Patel [15, 16].

These strata M_α are divisors only if g is a multiple of $d-1$, say $g = (d-1)k$ and then $\alpha = (k, k+1, \ldots, k+1, k+2)$. It is natural to ask for the divisor classes of the closures of these Maroni loci in the admissible cover compactification $\overline{\mathcal{H}}_{d,g}$ introduced by Harris and Mumford in [10]. This question was studied by Stankova–Frenkel in [18] for the trigonal case and for degrees 4 and 5 by Patel in his Harvard thesis [15]. Patel determined classes in a partial compactification of the Hurwitz space. Deopurkar and Patel calculated the class of the closure of the Maroni divisor in the trigonal case in [5].

In this article we consider the general case and we construct certain effective divisors containing the closure of the Maroni divisor and calculate their classes. We do this in a series of steps. We start by constructing a good model for the universal d-gonal map $\gamma : \mathcal{C} \to \mathbb{P}^1_{\mathcal{H}_{d,g}}$, where a good model stands for a proper flat and finite map $\tilde{\gamma} : Y \to \tilde{\mathbb{P}}$ that extends γ over a normalization $\widetilde{\mathcal{H}}_{d,g}$ of $\overline{\mathcal{H}}_{d,g}$. This allows us to extend the vector bundle V over $\mathbb{P}^1_{\mathcal{H}_{d,g}}$ used to define the Maroni invariant α to a reflexive sheaf over the compactification $\widetilde{\mathbb{P}}$ and ignoring codimension ≥ 3 loci we may consider it as a vector bundle. However, since the fibres of the projection $p : \widetilde{\mathbb{P}} \to \overline{\mathcal{H}}_{d,g}$ of the compactified space $\widetilde{\mathbb{P}}$ are no longer always isomorphic to $\mathbb{P}^1$, but can be chains of $\mathbb{P}^1$, we cannot directly extend the definition of α. Instead we extend the Maroni divisor as follows.

First we twist the extended vector bundle V by an explicit line bundle so that we get a vector bundle V' on $\widetilde{\mathbb{P}}$ which restricted to a general fibre of p is trivial. We associate a divisor class on $\widetilde{\mathbb{P}}$ to the global-to-local evaluation map

$$\mathrm{ev} : p^*p_*V' \longrightarrow V'$$

and show that this divisor class is represented by an effective divisor whose pullback under a section of $\widetilde{\mathbb{P}}$ gives an effective divisor $\mathfrak{m}_{\mathrm{st}}$ on $\widetilde{\mathcal{H}}_{d,g}$ that extends the Maroni divisor on $\mathcal{H}_{d,g}$. Our idea of using the morphism $p^*p_*V' \to V'$ was inspired by a paper of Brosius [2].

The vector bundle V (or rather reflexive sheaf) that we use is an extension of the dual of the cokernel of the natural map $\mathcal{O}_{\mathbb{P}^1_{\mathcal{H}_{d,g}}} \to \gamma_*\mathcal{O}_{\mathcal{C}}$. It is obtained by taking the structure sheaf on Y and taking the dual V of the cokernel of the natural map $\iota : \mathcal{O}_{\widetilde{\mathbb{P}}} \to \tilde{\gamma}_*\mathcal{O}_Y$ and by twisting in a standard way $V' = V \otimes M$, where M is an explicitly defined line bundle on $\widetilde{\mathbb{P}}$. With these choices our construction yields an explicit effective divisor class $\mathfrak{m}_{\mathrm{st}}$ that extends the Maroni divisor. We express the class $\mathfrak{m}_{\mathrm{st}}$ explicitly in terms of the classes of boundary divisors of the compactified Hurwitz space.

But we can replace $\mathcal{O}_Y$ by any effective line bundle $\mathcal{L}$ on Y that extends the trivial line bundle on $\mathcal{C}$ and instead take the dual $V_{\mathcal{L}}$ of the cokernel of the natural map $\iota : \mathcal{O}_{\widetilde{\mathbb{P}}} \to \tilde{\gamma}_*\mathcal{L}$. And then after twisting by M we can twist by an arbitrary line bundle N corresponding to a divisor with support on the boundary of $\widetilde{\mathbb{P}}$, that is, the singular fibres of the generic $\mathbb{P}^1$-fibration, as the result remains trivial on the generic fibre of p. Doing this produces variations $\mathfrak{m}_{\mathcal{L},N}$ of $\mathfrak{m}_{\mathrm{st}}$ each of which is a class of an effective divisor that extends the Maroni locus. In particular, with $\mathcal{L}$ trivial varying N allows us to get rid of the specific choice of M to make V trivial on the generic fibre.

We study the effect of changing $\mathcal{L}$ and N on the resulting effective divisors $\mathfrak{m}_{\mathcal{L},N}$. By appropriately choosing $\mathcal{L}$ and N we often can reduce the divisor $\mathfrak{m}_{\mathrm{st}}$ by removing superfluous divisors supported on the boundary. In fact, the effect of twisting is quadratic and we can find the critical points of this quadratic function. But these critical points correspond to elements in the rational Picard group and we thus have to approximate by integral points. If we take $\mathcal{L}$ trivial and vary N the result is a completely explicit combinatorial formula (Theorem 9.3) for this effective class in terms of the boundary classes of the compactified Hurwitz space and it involves only local contributions. If we vary also $\mathcal{L}$ it involves global classes (Theorem 10.4), but under certain assumptions the contributions still can be given explicitly (Theorem 11.3) and then finding the optimal choices of $(\mathcal{L}, N)$ reduces to a combinatorial problem.

The rational Picard group of the compactified Hurwitz space is conjectured to be generated by the classes of the irreducible boundary components. In any case, we work inside the subgroup generated by these boundary components. We can define an effective divisor containing the Maroni class by

$$\mathfrak{m}_{\min} = \cap_{\mathcal{L},N} \mathfrak{m}_{\mathcal{L},N} ,$$

where $\mathcal{L}$ and N vary. We do not know whether $\mathfrak{m}_{\min}$ coincides with the closure of the Maroni locus, but in the trigonal case ($d = 3$ and g even) it does: the class of the closure of the Maroni locus coincides with $\mathfrak{m}_{\mathcal{L},N}$ for suitable $(\mathcal{L}, N)$. In fact, in [5] Deopurkar and Patel determined the class of the Zariski closure of the Maroni divisor in the trigonal case and we show that an appropriate choice of $\mathcal{L}$ and N reproduces the class found by Deopurkar and Patel. We also show that the effective class found by Patel in [15] containing the Maroni locus on a partial compactification is in general larger than the class of the Zariski closure of the Maroni locus. We point out that we do not use the standard tool of test curves that seems inadequate here.

In a subsequent paper ([20]) we construct analogues of the Maroni stratifications for the case where the genus g is not divisible by $d - 1$ that possess strata that are divisors. The methods in this article extend to that case.

We hope that this method to determine the class of an effective divisor extending the Maroni divisor is of independent interest for studying cycle classes on moduli spaces.

We finish with a remark on the terminology. We will encounter reflexive sheaves, but as these are locally free outside a locus of codimension ≥ 3 on smooth spaces, the fact that these are not necessarily locally free will not affect our divisor class calculations and frequently we will treat these reflexive sheaves as if they were vector bundles. Finally, we will often use the same symbol for a divisor and its divisor class or cohomology class.

2 Maroni Loci

Let $\mathcal{H}_{d,g}$ be the Hurwitz space of genus g degree d simply-branched covers of $\mathbb{P}^1$ over the field of complex numbers with ordered branch points on $\mathbb{P}^1$. Here simply-branched means that every fibre contains at least $d - 1$ points. It is a Deligne–Mumford stack of dimension $2g+2d-5$. To a simply-branched cover $\gamma : C \to \mathbb{P}^1$ we can associate an exact sequence of sheaves

$$0 \to \mathcal{E}_1 \to \gamma_* \mathcal{O}_C \xrightarrow{\mathrm{tr}} \mathcal{O}_{\mathbb{P}^1} \to 0 ,$$

where tr stands for the trace map and $\mathcal{E}_1 = \ker(\mathrm{tr})$. Since tr is multiplication by the degree d on the pull back of $\mathcal{O}_{\mathbb{P}^1}$ under γ, the sequence splits and $\mathcal{E}_1$ is locally free as a direct summand of a locally free sheaf. Let $\mathcal{E} = \mathcal{E}_1^\vee$ be the dual locally free sheaf, which we view as a vector bundle V of rank $d-1$. As a bundle on $\mathbb{P}^1$ it can be written as a direct sum $\oplus_{i=1}^{d-1}\mathcal{O}(a_i)$ with $\sum_{i=1}^{d-1} a_i = g+d-1$. For a given cover γ we can look at the type, that is, the $(d-1)$tuple $\alpha = (a_1, \ldots, a_{d-1})$, where we may and will assume that $a_1 \le a_2 \le \cdots \le a_{d-1}$. Such a type defines a locus $\mathcal{M}_\alpha$ in $\mathcal{H}_{d,g}$, namely the Zariski closure of the set of covers γ whose associated V has type α. We are interested in the case where such a Maroni locus is a divisor. According to a result by Coppens and Martens (see [4]) and Patel [15, Theorems 1.13 and 1.15] we find that this happens if and only if the genus is a multiple of $d-1$, say

$$g = (d-1)\,k,$$

and that then generically we have $\alpha = (k+1, \ldots, k+1)$, that is, V is balanced, while for a point on this divisor generically we have $\alpha = (k, k+1, \ldots, k+1, k+2)$. Moreover, the divisor is irreducible. Thus for covers $\gamma : C \to \mathbb{P}^1$ with $g = (d-1)k$ not contained in the Maroni locus $\mathcal{M}_\alpha$ with $\alpha = (k, k+1, \ldots, k+1, k+2)$ the vector bundle V is balanced, while generically on the Maroni locus we have a minimal deviation from this behaviour, see [15, Theorem 1.15].

We are interested in the cycle classes of this Maroni divisor. By a $\mathbb{P}^1$-fibration we mean a morphism $p : \mathbb{P} \to X$, where $\mathbb{P}$ is the projectivization of a rank 2 vector bundle on X. In our case we have a $\mathbb{P}^1$-fibration $\mathbb{P}$ over our base together with a vector bundle V on $\mathbb{P}$. We may twist the vector bundle V by tensoring by $\mathcal{O}(-k-1)$, that is, by $\mathcal{O}((-k-1)S)$, with S the image of a section of the $\mathbb{P}^1$-fibration. Carrying out the above construction in families we arrive at the situation, where we have a vector bundle V of rank r on a $\mathbb{P}^1$-fibration $\mathbb{P}$ over $\mathcal{H}_{d,g}$ that is trivial on the generic fibre $\mathbb{P}^1$ and has minimal deviation $\mathcal{O}(-1) \oplus \mathcal{O} \oplus \cdots \mathcal{O} \oplus \mathcal{O}(1)$ on a divisor in $\mathcal{H}_{d,g}$ (and larger deviations in higher codimension only). This will imply that the support of R^1p_*V has codimension > 1.

So suppose that we have a $\mathbb{P}^1$-fibration $p : \mathbb{P} \to X$, with X a smooth base, and a vector bundle V of rank r (or more generally a reflexive sheaf) on $\mathbb{P}$ which is trivial on the generic fibre. We consider the canonical global-to-local map of sheaves on $\mathbb{P}$

$$\mathrm{ev} : p^*p_*V \longrightarrow V$$

given by evaluation. We note that p_*V is a reflexive sheaf (isomorphic to its double dual) by [11, Corollary 1.7]. Therefore this sheaf is locally free outside a locus of codimension ≥ 3. We will work outside this locus and then may assume that p_*V is locally free. If U is an open subset of X over which p_*V is free of rank r we can choose r generating sections $s_1, \dots, s_r$ of p_*V on U and consider their pullbacks p^*s_i to $\mathbb{P}$. The images of these s_i under the restriction map

$$p^*p_*V \to H^0(p^{-1}(x), V|p^{-1}(x))$$

to the fibre $p^{-1}(x) = \mathbb{P}^1$ over x for $x \in U$ generate a subspace of $H^0(\mathbb{P}^1, V|p^{-1}(x))$, but if $V|p^{-1}(x)$ has type $\oplus_{i=1}^r \mathcal{O}(a_i)$ with at least one $a_i < 0$ these sections cannot generate the stalk $\oplus_{i=1}^r \mathcal{O}(a_i)$, since these sections are located in the "non-negative part"

$$\oplus_{a_i \geq 0} \mathcal{O}(a_i)\,.$$

Therefore the global-to-local map $\mathrm{ev} : p^*p_*V \to V$, a map of vector bundles of the same rank, must have vanishing determinant. On the other hand if $V|p^{-1}(x)$ is trivial of rank r, then by Grauert's theorem the r sections p^*s_i, which are linearly independent in $H^0(p^{-1}(x), V|p^{-1}(x)) = H^0(\mathbb{P}^1, \mathcal{O}^r)$, necessarily generate $V|p^{-1}(x)$. We summarize as follows.

Proposition 2.1. Outside a locus of codimension ≥ 3 the support of the vanishing locus of the determinant of the homomorphism $\mathrm{ev} : p^*p_*V \to V$ coincides with the inverse image under p of the Maroni locus. □

We are interested in the first Chern class $c_1(Q) = c_1(V) - c_1(p^*p_*V)$ of the degeneracy locus Q of the evaluation map $\mathrm{ev} : p^*p_*V \to V$. Since the map ev is injective in codimension 1 this class coincides with the first Chern class of the cokernel of the map ev. This is a class on $\mathbb{P}$, but in accordance with Proposition 2.1 it is in fact a pull back class from X as we shall see now.

Lemma 2.2. If V is a vector bundle on a $\mathbb{P}^1$-fibration $p : \mathbb{P} \to X$ which is trivial on the generic fibre and has minimal deviation $(-1, 0, \dots, 0, 1)$ over a divisor in X, then the divisor class of the determinant of $\mathrm{ev} : p^*p_*V \to V$ is given by

$$c_1(Q) = c_1(V) - c_1(p^*p_*V) = p^*p_*c_2(V)\,. \qquad \square$$

Proof. The sheaf R^1p_*V has support in codimension ≥ 2, therefore we can neglect it for the calculation of $c_1(p_*V)$. Let θ_p be the first Chern class of the relative dualizing sheaf

ω_p of p. Then the Grothendieck–Riemann–Roch Theorem applied to p and V gives us

$$\begin{aligned} c_1(p_*V) &= p_*[\mathrm{ch}(V)\,\mathrm{td}(T_p)]_{(2)} \\ &= p_*[(r + c_1(V) + c_1^2(V)/2 - c_2(V))(1 - \theta_p/2 + \theta_p^2/12)]_{(2)} \\ &= p_*[r\,\theta_p^2/12 - c_1(V)\theta_p/2 + c_1^2(V)/2 - c_2(V)]\,. \end{aligned}$$

Now the class $c_1(V)$ restricted to the fibres is trivial (because of degree 0 on a $\mathbb{P}^1$), hence $c_1(V)$ is a pullback $c_1(p^*L)$ for some line bundle L on X. If $\sigma : X \to \mathbb{P}$ is a section with image S, then $\omega_p \cong \mathcal{O}(-2S) \otimes p^*\sigma^*\mathcal{O}(S)$, and therefore

$$\theta_p^2 = 4\,S^2 - 4\,S \cdot p^*\sigma^*S + p^*\sigma^*S^2\,.$$

We observe $p_*p^*\sigma^*S^2 = 0$, $S^2 = \sigma_*\sigma^*S$, $S \cdot p^*\sigma^*S = \sigma_*\sigma^*p^*\sigma^*S$ by the projection formula, and so $\sigma_*\sigma^*p^*\sigma^*S = \sigma_*\sigma^*S$ because $p\,\sigma = 1$. So $p_*\theta_p^2 = 4\,\sigma_*\sigma^*S - 4\,\sigma_*\sigma^*S = 0$. Furthermore, we have

$$p_*(c_1(V) \cdot \theta_p) = p_*(p^*c_1(L) \cdot (-2\,S + p^*\sigma^*S)) = -2\,c_1(L) \cdot p_*S = -2\,c_1(L)\,.$$

We conclude that $c_1(p_*V) = c_1(L) - p_*c_2(V)$, hence $[\det(\mathrm{ev})] = c_1(V) - p^*c_1(p_*V) = p^*p_*c_2(V)$. ■

Corollary 2.3. The pullback of $c_1(Q)$ under a section $\sigma : X \to \mathbb{P}$ of p gives an effective divisor with support on the locus over which V is not trivial. Its class is equal to $p_*c_2(V)$. □

Proof. The pullback is $\sigma^*p^*p_*c_2(V) = p_*c_2(V)$. ■

If we do not assume that R^1p_*V is trivial in codimension 1 then we can reformulate the result as follows.

Lemma 2.4. If V is a vector bundle on a $\mathbb{P}^1$-fibration $p : \mathbb{P} \to X$ which is trivial on the generic fibre, then we have the identity

$$c_1(Q) + p^*c_1(R^1p_*V) = p^*p_*c_2(V)\,. \qquad \square$$

Since in general we have a bundle V on a $\mathbb{P}^1$-fibration which is balanced of type $(k, \ldots, k)$ on the generic fibre, but not trivial on the generic fibre, we check what happens if we twist by a line bundle that on the generic fibre is a power of $\mathcal{O}_{\mathbb{P}^1}(1)$.

The canonical global-to-local map $p^*p_*(V(-k)) \to V(-k)$ induces now a canonical map $(p^*p_*V(-k))(k) \to V$ with degeneracy locus Q.

Lemma 2.5. If V is a vector bundle of rank r on a $\mathbb{P}^1$-fibration $p : \mathbb{P} \to X$ which is balanced on the generic fibre and has minimal deviation $(k-1, k, \ldots, k, k+1)$ over a divisor in X, then the pull back of the first Chern class of the degeneracy locus Q of $\mathrm{ev} : (p^*p_*V(-k))(k) \to V$ under σ^* is given by the effective class

$$c_1(Q) = p_*\left(c_2(V) - \frac{r-1}{2\,r} c_1^2(V)\right). \tag{1}$$

□

Proof. Let $c_1(\mathcal{O}(-k)) = -kS$ with S the image of a section. We have

$$c_2(V(-k)) = c_2(V) + (r-1)c_1(V)\cdot(-kS) + \frac{r(r-1)}{2}k^2S^2\,.$$

Since $p_*c_1^2(V(-k)) = 0$ we get

$$p_*c_1^2(V) = 2\,r\,p_*(c_1(V)\cdot kS) - r^2\,p_*(k^2S^2)\,.$$

The formula follows. The effectivity follows by Proposition 2.1 ■

Remark 2.6. The expression $c_2(V) - ((r-1)/2r)\,c_1^2(V)$ that appears in formula (1) is the same one as in Bogomolov's result on the Chern numbers of a semi-stable vector bundle on a surface, cf. [8, Thm 0.3] □

3 Extension to Generic $\mathbb{P}^1$-Fibrations

Since we want to extend the calculation of the class of a Maroni divisor to the full space $\overline{\mathcal{H}}_{d,g}$ of admissible covers, we have to deal with fibre spaces which are generically a $\mathbb{P}^1$-fibration but over a divisor have singular fibres which are chains of smooth rational curves $P_0, P_1, \ldots, P_n$. We call these admissible generic $\mathbb{P}^1$-fibrations. We shall see later where such fibrations result from.

For defining the Maroni locus we will have to work over a base space which is the compactified Hurwitz space, while for most of the later calculations it will suffice to deal with the case of a 1-dimensional complete base curve B and a generic $\mathbb{P}^1$-fibration. Anyway, to see what is going on, it might help the reader to assume that we are dealing with the case where the base B is one dimensional; in such a case we are dealing with surfaces and surface singularities and in a singular fibre the two extremal curves P_0 and P_n of the chain have self-intersection number -1, while the remaining P_i have $P_i^2 = -2$. Our general case is locally a product of such a possibly singular surface times smooth affine space.

Let now $p : \Pi \to B$ be an admissible generic $\mathbb{P}^1$-fibration over a base curve B and V be a vector bundle with trivial fibre $\mathcal{O}_{\mathbb{P}^1}^r$ on the generic fibre $\mathbb{P}^1$ of p. For such a vector bundle V over Π we can consider the global-to-local map

$$p^* p_* V \longrightarrow V\,.$$

We know that on the open set U over which p is a $\mathbb{P}^1$-fibration, the support of the degeneracy locus Q outside a codimension ≥ 3 locus is given by the vanishing of the determinant. We want to analyse Q near special fibres.

Even if we do not know the behaviour of V on the singular fibres of the generic $\mathbb{P}^1$-fibration, we can estimate the first Chern class of Q in the following way.

Let V be a vector bundle of rank r on a generic $\mathbb{P}^1$-fibration $p : \Pi \to B$ over a smooth base B. We assume that V is trivial of rank r on the generic fibre. The first Chern class of V restricted to a smooth fibre is trivial since it is of degree 0 on $\mathbb{P}^1$, hence we can write

$$c_1(V) = p^* D + A, \tag{2}$$

with D a divisor class on B and A a divisor class supported on the singular fibres of p. (We denote the divisor and its class by the same symbol.) Again we have a canonical morphism

$$\mathrm{ev} : p^* p_* V \longrightarrow V\,.$$

The analogue of the formula (1) in Lemma 2.5 is the following.

Proposition 3.1. Let V be a vector bundle of rank r on the admissible generic $\mathbb{P}^1$-fibration Π over B which is trivial on the generic fibre and with $c_1(V) = p^* D + A$ for D a divisor class on B and A a divisor class supported on the singular fibres. The first Chern class of the degeneracy locus Q of the canonical global-to-local map $p^* p_* V \to V$ is given by the formula

$$c_1(Q) + p^* c_1(R^1 p_* V) = A + p^* p_* \left(\frac{1}{2} A \cdot \theta_p - \frac{1}{2\,r} A^2 \right) + p^* p_*(c_2(V))\,. \qquad \square$$

Remark 3.2. The expression $A + p^*p_*((1/2)\,A\cdot\theta_p - (1/2r)\,A^2)$ does not change if we replace A by $A+p^*D$ with D a divisor (class) on B. Indeed, for any divisor D on B one has

$$p_*(p^*D\cdot\theta_p) = -2\,D\,, \quad A\cdot p^*D = 0, \quad \text{and} \quad (p^*D)^2 = 0\,. \qquad \square$$

Proof. We apply Grothendieck–Riemann–Roch to p and V:

$$\begin{aligned} c_1(p_*V - R^1p_*V) &= p_*(\mathrm{ch}(V)\cdot\mathrm{td}(T_p))_{(2)} \\ &= p_*\left[\left(r + c_1(V) + \frac{1}{2}(c_1^2(V) - 2\,c_2(V))\right)\left(1 - \frac{1}{2}\theta_p + \mathrm{td}_2(T_p)\right)\right]_{(2)} \\ &= p_*\left(r\,\mathrm{td}_2(T_p) - \frac{1}{2}c_1(V)\cdot\theta_p + \frac{1}{2}c_1^2(V) - c_2(V)\right)\,. \end{aligned}$$

Using the identities in Remark 3.2 one sees that

$$p_*c_1^2(V) = p_*(A^2) \quad \text{and} \quad p_*(c_1(V)\cdot\theta_p) = -2\,D + p_*(A\cdot\theta_p)\,.$$

We claim that $p_*(\mathrm{td}_2(T_p)) = 0$. For this one applies Grothendieck–Riemann–Roch to p and $\mathcal{O}_\Pi$. Note that $p_*\mathcal{O}_\Pi = \mathcal{O}_Y$ and $R^1p_*\mathcal{O}_\Pi = (p_*\omega_p)^\vee = (0)$, so we get $0 = c_1(\mathcal{O}_Y) = p_*(\mathrm{td}_2(T_p))$. ■

Just as above, if we start with a vector bundle V that is balanced on the generic fibre and if there exists a line bundle M on Π such that $V' = V\otimes M$ is trivial of rank r on the generic fibre, then in the formula the second Chern class gets adapted:

Corollary 3.3. Let V be a vector bundle on Π that is balanced on the generic fibre of Π over B and M a line bundle on Π such that $V' = V\otimes M$ is trivial of rank r on the generic fibre of Π over B. We denote the degeneracy locus of the natural map $p^*p_*V' \to V'$ by Q. If we write as before $c_1(V') = p^*D + A$ with A supported on the singular fibres, we have

$$\begin{aligned} c_1(Q) + p^*c_1(R^1p_*V') = A + p^*p_*\left(\frac{1}{2}A\cdot\theta_p - \frac{1}{2\,r}A^2\right) \\ + p^*p_*\left(c_2(V) - \frac{r-1}{2\,r}c_1^2(V)\right). \end{aligned} \qquad \square$$

Now we consider the effectivity of the classes occurring here.

Proposition 3.4. The expression $c_1(Q) + p^*c_1(R^1p_*V')$ is an effective divisor class. $\square$

Proof. We have that $c_1(Q)$ is effective as the class of a degeneracy locus. The stalk of the sheaf R^1p_*V' over a general point is trivial as V' is trivial over the fibre over that point. Therefore R^1p_*V' is a torsion sheaf. The determinant of a torsion sheaf admits a non-trivial regular section, see [12, Proposition 5.6.14], or alternatively, apply the Grothendieck–Riemann–Roch theorem to the embedding of the support of R^1p_*V' and the sheaf R^1p_*V' and get that $c_1(R^1p_*V')$ is represented by a positive multiple (the rank) of the codimension 1-cycle of the support. Therefore the class $c_1(R^1p_*V')$ is an effective class. ■

We conclude two things from the above discussion:

(1) The class $c_1(Q) + p^*c_1(R^1p_*V') - A$ is a pullback class under p^*;
(2) The class $c_1(Q) + p^*c_1(R^1p_*V')$ is an effective class.

The first follows from Corollary 3.3 and the second by Proposition 3.4. But we want a class that is both effective and a pullback under p. We will do this by adding (or subtracting) a pullback to $c_1(Q) + p^*c_1(R^1p_*V') - A$ so that it becomes effective in a minimal way. We shall use the following lemma.

Lemma 3.5. Suppose that D is an effective divisor on Π which is linearly equivalent to a pullback p^*T from the one-dimensional base B. If $F_0 = \sum_{i=0}^n P_i$ is a reduced fibre and if D contains one irreducible component P_i with multiplicity η then it contains all irreducible components P_i with multiplicity η. □

Proof. Observe that D cannot contain horizontal components since this would violate $D \cdot F = p^*T \cdot F = 0$ for a fibre F. But the only solutions of $\sum_j \alpha_j P_j \cdot P_i = 0$ for all $i = 0, \dots, n$ are the multiples of the fibre F_0. ■

We now consider the situation over a possibly higher dimensional base.

Definition 3.6. Suppose that G is a divisor on Π supported in the singular fibres of p and suppose that its image $p(G)$ is supported on an irreducible divisor D with $p^{-1}(D) = \sum_{i=0}^n E_i$ with E_i irreducible. We assume that in a fibre over a general point x of D the components $E_0, \dots, E_n$ give rise to the components $P_0, \dots, P_n$ of $p^{-1}(x)$ forming a chain of rational curves. Write $G = \sum_{i=0}^n e_iE_i$. We then set

$$G_{\text{sh}} := \sum_{i=0}^{n} (e_i - e_{\max})E_i\,,$$

where $e_{\max} := \max\{e_i : i = 0, \ldots, n\}$. By doing this for all divisors G (not necessarily with irreducible image under p) and extending it linearly we associate to any divisor G with support in the singular fibres a (shifted) divisor G_{sh}. □

The effect of this is shifting the coefficients e_i by a common constant such that the maximum coefficient becomes 0.

Definition 3.7. For a divisor G as in Definition 3.6 we let F_G be the divisor that is a sum of fibre divisors, each fibre divisor $F = \sum_{i=1}^{n} E_i$ coming with multiplicity $\min\{\mathrm{ord}_{E_i}(G) : i = 1, \ldots, n\}$. We have $F_G = G + (-G)_{\mathrm{sh}}$. □

Now we improve our divisor by adding $F_A = (-A)_{\mathrm{sh}} - (-A)$.

Proposition 3.8. If $c_1(V)$ modulo a pull back of a divisor under p is linearly equivalent to a divisor A supported on the singular fibres of Π such that $A = \sum_D A^D$ with D running through irreducible divisors D on the base B such that $p^{-1}D$ is a chain $\sum_{i=1}^{n_D} E_i^D$ and $A^D = \sum_i e_i^D E_i^D$ then the expression

$$c_1(Q) + p^* c_1(R^1 p_* V') + (-A)_{\mathrm{sh}}$$

is linearly equivalent to the pull back under p of an effective class on B. □

Proof. We set $\Gamma = c_1(Q) + p^* c_1(R^1 p_* V') - A$. We may consider one irreducible divisor D on the base B such that $p^{-1}(D) = \sum_{i=0}^{n} E_i$ is as in Definition 3.6. If $-A = \sum e_i E_i$ and all $e_i \geq 0$, then all numbers $\mathrm{ord}_{E_i}\Gamma$ are non-negative and by Lemma 3.5 the divisor Γ contains the whole fibre $F = \sum_{i=1}^{n} E_i$ with multiplicity $\geq e_{\max} = \max\{e_i : i = 1, \ldots, n\}$ and we can then subtract it from Γ while still keeping an effective divisor class which is a pull back.

Suppose then that $-A = \sum_{i=1}^{n} e_i E_i$ is not effective. We can replace $-A$ modulo pull backs from the base by the effective class

$$-A' = -A + |e_{\min}| F$$

with $e_{\min} = \min\{e_i : i = 1, \ldots, n\}$. Then using Lemma 3.5 we can subtract from $c_1(Q) + p^* c_1(p_* V') - A'$ a multiple ϵF of F with $\epsilon = \max\{e_i + |e_{\min}| : i = 1, \ldots, n\}$. So in total with $-A = \sum_{i=1}^{n} e_i E_i$ we can replace $-A$ in $c_1(Q) + p^* c_1(p_* V') - A$ by

$$-A + |e_{\min}| F - (e_{\max} + |e_{\min}|) F = \sum_{i=1}^{n} (e_i - e_{\max}) E_i$$

in order to retain an effective divisor $c_1(Q) + p^* c_1(R^1 p_* V') + (-A)_{\mathrm{sh}}$. Doing this for all divisors D with support on the singular fibres we replace $-A$ by $(-A)_{\mathrm{sh}}$. ■

Remark 3.9. If all the coefficients of $-A = \sum_{i=1}^{n} e_i E_i$ are strictly negative, then we add something in order to get an effective divisor. On the other hand if $-A$ has a non-negative coefficient then not only is $c_1(Q) - p^* c_1(R^1 p_* V') - A$ an effective divisor (cf. Lemma 3.5), but we can improve it, that is, find a smaller effective divisor. □

Using Remark 3.2 we find the following result.

Theorem 3.10. Suppose that V is a vector bundle on Π of rank r that is balanced on the generic fibre of p. With A as in Corollary 3.3 we find that

$$F_A + p^* p_* \left(\frac{1}{2} A \cdot \theta_p - \frac{1}{2r} A^2 \right) + p^* p_* \left(c_2(V) - \frac{r-1}{2r} c_1^2(V) \right),$$

where θ_p is the first Chern class of the relative dualizing sheaf ω_p of p, is represented by an effective divisor class which is pullback under p^*. □

4 A Good Model

Let $\mathcal{H}_{d,g}$ be the Hurwitz space of covers $C \to \mathbb{P}^1$ of degree d and genus g. It can be extended to the space $\overline{\mathcal{H}}_{d,g}$ of admissible covers of degree d and genus g. The "boundary" $\overline{\mathcal{H}}_{d,g} - \mathcal{H}_{d,g}$ is a sum of finitely many divisors $S_{j,\mu} = S_{b-j,\mu}$ with $b = 2g - 2 + 2d$ indexed by $2 \le j \le b-2$ and a partition $\mu = (m_1, \ldots, m_n)$ of d. In general the divisor $S_{j,\mu}$ will be reducible; a generic point of a component corresponds to an admissible cover $C \to P$ with P a curve of genus 0 having two components P_1 and P_2 intersecting in one point Q such that P_1 (respectively, P_2) has $j_1 = j$ or $j_1 = b - j$ (respectively, j_2) branch points with $j_1 + j_2 = b$ and the inverse image of Q consists of n points $Q_1, \ldots, Q_n$ with ramification indices $m_1, \ldots, m_n$.

This space $\overline{\mathcal{H}}_{d,g}$ is not normal, and therefore we consider the normalization $\widetilde{\mathcal{H}}_{d,g}$ of $\overline{\mathcal{H}}_{d,g}$. This is now a smooth stack. Over $\widetilde{\mathcal{H}}_{d,g}$ we then have a universal curve $\varpi : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{H}}_{d,g}$ in the sense of stacks.

Our goal is to extend the vector bundle on $\mathbb{P}^1_{\mathcal{H}_{d,g}}$ to a vector bundle on a compactification. For this we must extend our universal d-gonal cover and our first aim here is to construct a good model for the universal d-gonal map

$$\gamma : \mathcal{C} \to \mathbb{P}^1_{\mathcal{H}_{d,g}}.$$

By a good model we mean a proper flat map $\tilde{\gamma} : Y \to \widetilde{\mathbb{P}}$ that extends γ over $\widetilde{\mathcal{H}}_{d,g}$.

Recall that the universal curve $\widetilde{\mathcal{C}}$ over $\widetilde{\mathcal{H}}_{d,g}$ fits into a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "c"] \arrow[d, "\varpi"'] & \overline{\mathcal{M}}_{0,b+1} \arrow[d, "\pi_{b+1}"] \\ \widetilde{\mathcal{H}}_{d,g} \arrow[r, "h"] & \overline{\mathcal{M}}_{0,b} \end{tikzcd}$$

where $\mathcal{M}_{0,b}$ is the moduli space of stable b-pointed curves of genus 0 and π_{b+1} is the map that forgets the $(b+1)$th point. We let $\mathbb{P}$ be the fibre product of $\overline{\mathcal{M}}_{0,b+1}$ and $\widetilde{\mathcal{H}}_{d,g}$ over $\overline{\mathcal{M}}_{0,b}$. We find a diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\alpha"] \arrow[dr, "\varpi"'] & \mathbb{P} \arrow[r, "c'"] \arrow[d, "\varpi'"] & \overline{\mathcal{M}}_{0,b+1} \arrow[d, "\pi_{b+1}"] \\ & \widetilde{\mathcal{H}}_{d,g} \arrow[r, "h"] & \overline{\mathcal{M}}_{0,b} \end{tikzcd}$$

At this point we consider $\widetilde{\mathcal{H}}_{d,g}$ as our base B. For later use we point out that normalization commutes with smooth base change. This gives a composition of morphisms $\widetilde{\mathcal{C}}_B \to \mathbb{P}_B \to B$.

Lemma 4.1. The spaces $\widetilde{\mathcal{C}}_B$ and $\mathbb{P}_B$ both have at most A_k singularities. □

Proof. For the proof we may consider a one-dimensional base B. Locally at a non-smooth point of ϖ the map π_{b+1} is given by $t = uv$, while the map ϖ is given by $x_\nu y_\nu = t_\nu$ and c by $u = x_\nu^{m_\nu}$ and $v = y_\nu^{m_\nu}$ in suitable local coordinates before we normalize the Hurwitz space. That normalization has the effect of replacing t by s^m with $m = \mathrm{lcm}(m_1, \ldots, m_n)$. After pulling back π to $\widetilde{\mathcal{H}}_{d,g}$ we have local equations $s^m = uv$, $x_\nu y_\nu = s^{m/m_\nu}$ and still $u = x_\nu^{m_\nu}$ and $v = y_\nu^{m_\nu}$. So the local equations of $\mathbb{P}$ over $\widetilde{\mathcal{H}}_{d,g}$ are $s^m = uv$ and this creates A_{m-1} singularities. In turn for $\widetilde{\mathcal{C}}$ we then find local equations at the nodes of the form $x_\nu y_\nu = s^{m/m_\nu}$ which are singularities of type $A_{m/m_\nu - 1}$. ■

We now will work over B and will suppress the index B. Since $\mathbb{P}$ is not smooth we resolve its rational singularities and find a model $p : \widetilde{\mathbb{P}} \to \mathbb{P}$ resolving the singularities in a minimal way and take the fibre product of $\widetilde{\mathbb{P}}$ and $\widetilde{\mathcal{C}}$ over $\mathbb{P}$:

$$Y := \text{normalization of } \widetilde{\mathbb{P}} \text{ in } \mathbb{C}(\widetilde{\mathcal{C}}) \quad \text{and} \quad \widetilde{Y} := \text{ the resolution of } Y\,.$$

This fits into the following commutative diagram

Diagram 4.2.

$$\begin{tikzcd}
\widetilde{Y} \arrow[dr, "\nu"] \arrow[ddr, "\tilde{\pi}"'] \arrow[drr, bend left, "\rho"] & & & \\
& Y \arrow[r] \arrow[d, "\pi"] & \widetilde{\mathcal{C}} \arrow[d] \arrow[ddr, "\varpi"] & \\
& \widetilde{\mathbb{P}} \arrow[r] \arrow[drr, "p"'] & \mathbb{P} \arrow[dr] & \\
& & & B
\end{tikzcd}$$

We denote by $q : \widetilde{Y} \to B$ the composition from upper left to lower right. □

Proposition 4.3. The map $\pi : Y \to \widetilde{\mathbb{P}}$ is a flat map. □

Proof. We use the fact that if $f : A \to B$ is a finite morphism with A Cohen–Macaulay (actually proven in Lemma 5.1 later on) and B smooth then f is flat, see [9, 6.1.5]. The map $\widetilde{\mathcal{C}} \to \mathbb{P}$ is finite, hence the normalization of its base change too. ■

The fact that π is a flat map enables us to extend V as a vector bundle. However, as we shall see it will suffice to work with reflexive sheaves on smooth spaces.

Now we look at the direct image sheaves $R^i q_* \mathcal{O}_{\widetilde{Y}}$. By a spectral sequence argument one gets the following lemma, the proof of which is left to the reader.

Lemma 4.4. Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\mathcal{F}$ a coherent sheaf on X with $R^j f_* \mathcal{F} = (0)$ for all $j \geq 1$. Then we have $R^i (g \circ f)_* \mathcal{F} = R^i g_* (f_* \mathcal{F})$. □

We apply this first to $\tilde{\pi} = \pi \circ \nu$ and $\mathcal{F} = \mathcal{O}_{\widetilde{Y}}$ which has $R^i \nu_* \mathcal{O}_{\widetilde{Y}} = (0)$ for $i \geq 1$, since we resolved rational singularities only, so that $\tilde{\pi}_* \mathcal{O}_{\widetilde{Y}} = \pi_* \mathcal{O}_Y$ and then observe that

$$R^i \tilde{\pi}_* \mathcal{O}_{\widetilde{Y}} = R^i \pi_* (\nu_* \mathcal{O}_{\widetilde{Y}}) = R^i \pi_* \mathcal{O}_Y = (0) \qquad \text{for} \qquad i \geq 1$$

by the finiteness of π. Then we apply the same argument again to the map $q = p \circ \tilde{\pi}$ and we find:

Lemma 4.5. We have $R^i q_* \mathcal{O}_{\widetilde{Y}} = R^i p_* (\pi_* \mathcal{O}_Y)$ for $i = 0$ and $i = 1$. Moreover, we have $p_* (\pi_* \mathcal{O}_Y) = \mathcal{O}_B$. □

Proof. The first statement follows directly from the above. Furthermore, we have $p_*(\pi_*\mathcal{O}_Y) = q_*\mathcal{O}_{\widetilde{Y}} = \varpi_*(\rho_*\mathcal{O}_{\widetilde{Y}}) = \mathcal{O}_B$. ■

Corollary 4.6. The restriction of $\pi_*\mathcal{O}_Y$ to the generic fibre of p is equal to $\mathcal{O} \oplus \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_{d-1})$ with $a_i \geq 1$ satisfying $\sum_{i=1}^{d-1} a_i = g + d - 1$. □

Proof. By Riemann–Roch the restriction of $\pi_*\mathcal{O}_Y$ to a fibre of p has degree $-(g+d-1)$. This implies that the restriction to the generic fibre P, which is a $\mathbb{P}^1$, is $\oplus_{i=0}^{d-1}\mathcal{O}(b_i)$ with $b_i \geq b_{i+1}$ for $i = 0, \ldots, d-2$ and $\sum b_i = -g - d + 1$. Since $q_*\mathcal{O}_{\widetilde{Y}}$ equals $\mathcal{O}_B$ we find that $b_0 = 0$ and $b_i < 0$ for $i \geq 1$. We put $a_i = -b_i$ and get the result. ■

5 Local Description of the Map $\tilde{\pi} : \widetilde{Y} \to \widetilde{\mathbb{P}}$

We need an explicit description of the singularities of Y and their resolution. We consider the central part of the diagram 4.2

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \widetilde{\mathcal{C}} \arrow[d, "c"] \\ \widetilde{\mathbb{P}} \arrow[r, "\beta"] & \mathbb{P} \end{tikzcd}$$

over our base B. We analyse the situation near a point s of B that is a general point of an irreducible component of the boundary divisor $S_{j,\mu}$. For the description of the situation we may restrict to a one-dimensional base B and deal with a surface over B. The general situation is locally isomorphic to the product of such a surface times affine space.

In the following the indices j and μ will be fixed and therefore dropped from the notation. For a partition μ of d we use the notation

$$\mu = (m_1, \ldots, m_n) \qquad \text{and} \quad m = m(\mu) := \operatorname{lcm}(m_1, \ldots, m_n)\,.$$

Over our point s the space $\mathbb{P}$ has a singularity τ which is a node, locally isomorphic to a quotient singularity $\mathbb{C}^2/(\mathbb{Z}/m\mathbb{Z})$ with action $(z_1, z_2) \mapsto (\zeta_m z_1, \zeta_m^{-1} z_2)$ with ζ_m a primitive mth root of unity. Analytically it is isomorphic to

$$A_\tau := \operatorname{Spec}(\mathbb{C}[u, v, s]/(s^m - uv))\,.$$

The cover $\widetilde{\mathcal{C}}$ of $\mathbb{P}$ has n points Q_ν ($\nu = 1, \ldots, n$) lying over our node τ with $\widetilde{\mathcal{C}}$ at the node Q_ν analytically given by the ring $\mathbb{C}[x_\nu, y_\nu, s]/(s^{m/m_\nu} - x_\nu y_\nu)$ with the map locally given by

$$u = x_\nu^{m_\nu},\ v = y_\nu^{m_\nu},\ s = s\,.$$

We consider the resolution $\widetilde{\mathbb{P}}$ of $\mathbb{P}$. It is obtained by gluing m copies Z_i ($i = 0, \ldots, m-1$) of $\mathbb{C}^2$ with coordinates (ξ_i, η_i) via

$$\xi_{i+1} = \eta_i^{-1},\ \eta_{i+1} = \xi_i \eta_i^2,$$

wherever this makes sense. Moreover, the map $Z_i \to \mathbb{P}$ is locally given by

$$u = \xi_i^{i+1}\eta_i^i,\ v = \xi_i^{m-i-1}\eta_i^{m-i},\ s = \xi_i\eta_i\,.$$

The exceptional divisor is a chain of $m-1$ smooth rational curves $E_1, \ldots, E_{m-1}$ with E_i given by the equation $\xi_{i-1} = 0$ in Z_{i-1} for $i = 1, \ldots, m-1$, or equivalently by $\eta_i = 0$ in Z_i. The proper transform of the two $\mathbb{P}^1$'s given by $\eta_0 = 0$ (respectively, by $\xi_{m-1} = 0$) lies in Z_0 and intersects E_1 (respectively, lies in Z_{m-1} and intersects E_{m-1}) transversally.

Let $\tilde{A}_\tau = \mathbb{C}[\xi_i, \eta_i]$ be the coordinate ring of Z_i. The inclusion $A_\tau \subset \tilde{A}_\tau$ corresponding locally to the map β is given by $u = \xi_i^{i+1}\eta_i^i$, $v = \xi_i^{m-i-1}\eta_i^{m-i}$, $s = \xi_i\eta_i$. Locally analytically Y is given by the normalization of the ring $\tilde{A}_\tau = \mathbb{C}[\xi_i, \eta_i]$ in the quotient field L of the ring $\mathbb{C}[x_\nu, y_\nu, s]/(s^{m/m_\nu} - x_\nu y_\nu)$. This latter ring locally analytically describes $\widetilde{C}$ near our point and this field L is given by $\mathbb{C}(x_\nu, s)$ and we have $u = x_\nu^{m_\nu}$. The inclusion $\tilde{A}_\tau \hookrightarrow L$ is given by $\xi_i = x_\nu^{m_\nu}/s^i$, $\eta_i = s^{i+1}/x_\nu^{m_\nu}$. The normalization that we want is given by the following lemma.

Lemma 5.1. The normalization is locally given by the normalization of the coordinate ring $R = \mathbb{C}[\xi_i, \eta_i, x_\nu]/(x_\nu^{m_\nu} - \xi_i^{i+1}\eta_i^i)$. □

Proof. Let N be the normalization of A_τ in L. The ring R is embedded (in the same way as $\tilde{A}_\tau$) in the field L and contains A_τ. Observe that $s = \xi_i\eta_i \in A_\tau \subseteq R$ and $x_\nu \in R$ and therefore L is the quotient field of R. Since $x_\nu^{m_\nu} = \xi_i^{i+1}\eta_i^i$, we have that $x_\nu \in N$. Therefore $A_\tau \subseteq R \subseteq N$ and N and A_τ and R have the same normalization. ■

The surface with coordinate ring R has one singularity which is a quotient singularity of type (n_i, q_i), that is, isomorphic to the quotient of $\mathbb{C}^2$ by the action $(z_1, z_2) \mapsto (\zeta_{n_i} z_1, \zeta_{n_i}^{q_i} z_2)$ with ζ_{n_i} a primitive n_ith root of unity.

To see which singularity this gives we replace the equation by

$$x_\nu^{n_i} = \xi_i^{\alpha_i}\eta_i^{\beta_i}$$

with

$$n_i = \frac{m_\nu}{\gcd(m_\nu, i(i+1))}, \; \alpha_i = \frac{i+1}{\gcd(m_\nu, i+1)}, \; \beta_i = \frac{i}{\gcd(m_\nu, i)} \, .$$

and find a quotient singularity of type (n_i, q_i) with $q_i = -\beta_i/\alpha_i \in (\mathbb{Z}/n_i\mathbb{Z})^*$.

Now we return to the general case where B is the normalized Hurwitz space $\widetilde{\mathcal{H}}_{d,g}$. Applying what we found above in the case at hand, with (j, μ) fixed, results on $\widetilde{\mathbb{P}}$ in a chain of exceptional divisors

$$E_1, \ldots, E_{m-1} \, .$$

Above E_i we find n divisors $T_{1,i}, \ldots, T_{n,i}$ with $T_{\nu,i}$ such that when we restrict to a fibre over general x we find that $T_{\nu,i}$ gives rise to a curve $T_{\nu,i}^{(x)}$ that maps to the exceptional curve $E_i^{(x)}$ with degree

$$d_{\nu,i} := \gcd(m_\nu, i)$$

and ramification degree $m_\nu/\gcd(m_\nu, i)$.

Corollary 5.2. The branch divisor W of the map $\tilde{\pi} : \tilde{Y} \to \widetilde{\mathbb{P}}$ (over $\widetilde{\mathcal{H}}_{d,g}$) consists of two disjoint parts: the sum $W_S = \Xi$ of the $b = 2g - 2 + 2d$ sections Ξ_i $(i = 1, \ldots, b)$ of p and a contribution W_E from the exceptional divisors of the map $\widetilde{Y} \to Y$ given by

$$W_E := \sum_{i=1}^{m-1} \left(\sum_{\nu=1}^{n} (m_\nu - d_{\nu,i}) \right) E_i \, . \qquad \square$$

6 Extending Our Vector Bundle

We want to extend the vector bundle V that is the dual of the kernel of the trace map $\mathrm{tr} : \gamma_* \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathbb{P}^1_{\mathcal{H}_{d,g}}}$ to a vector bundle on the normalization of the compactified Hurwitz space $\widetilde{\mathcal{H}}_{d,g}$. We shall assume that $d-1$ divides g and set

$$g = (d-1)k \, .$$

The general theory (see [15, Theorem 1.15]) will then tell us that the locally free sheaf V will be balanced on the general fibre P of p:

$$V_{|P} \cong \mathcal{O}_P(k+1)^{d-1} \, .$$

Note that $\tilde{\pi} : \tilde{Y} \to \widetilde{\mathbb{P}}$ generically is a degree d cover of smooth varieties. Then we take a line bundle $\mathcal{L}$ on $\widetilde{Y}$ that is trivial when restricted to the "interior" $q^{-1}(\mathcal{H}_{d,g})$:

$$\mathcal{L} = \mathcal{O}_{\widetilde{Y}}(Z),$$

with Z an effective divisor supported on the boundary. Since Z is effective we have an inclusion $\mathcal{O}_{\widetilde{Y}} \subset \mathcal{O}_{\widetilde{Y}}(Z)$ and we thus get an inclusion $\tilde{\pi}_*\mathcal{O}_{\widetilde{Y}} \subset \tilde{\pi}_*\mathcal{O}_{\widetilde{Y}}(Z)$ and since $\mathcal{O}_{\widetilde{\mathbb{P}}} \subset \tilde{\pi}_*\mathcal{O}_{\widetilde{Y}}$ we get an injective homomorphism

$$\iota = \iota_{\mathcal{L}} : \mathcal{O}_{\widetilde{\mathbb{P}}} \hookrightarrow \tilde{\pi}_*\mathcal{L}\,.$$

The dual $K_{\mathcal{L}}^{\vee}$ of the cokernel $K_{\mathcal{L}}$ of ι is a reflexive sheaf since it is the dual of a coherent sheaf and since we are working on smooth spaces by neglecting an algebraic subset of codimension ≥ 3, we may and will assume that it is locally free (see [11, Corollary 1.4]). Its restriction to $p^{-1}(\mathcal{H}_{d,g})$ is isomorphic to the bundle V considered before. We will denote this rank $d-1$ bundle $K_{\mathcal{L}}^{\vee}$ on $\widetilde{\mathbb{P}}$ by

$$V_{\mathcal{L}} := K_{\mathcal{L}}^{\vee}\,.$$

The choice of Z will give us freedom that we shall use later. But we start by assuming that Z is trivial, that is, we start by assuming

$$\mathcal{L} = \mathcal{O}_{\widetilde{Y}}\,.$$

We then find a rank $d-1$ bundle denoted by $V = V_{\mathcal{O}_{\widetilde{Y}}}$. In fact, by the results of Section 4 this special V is not only a reflexive sheaf but actually a vector bundle.

Lemma 6.1. We have

$$c_1(V) = -c_1(\tilde{\pi}_*\mathcal{O}_{\widetilde{Y}}) = W/2\,. \qquad \square$$

Proof. The first equality follows directly from the definition, while the second follows by applying Grothendieck–Riemann–Roch to $\tilde{\pi}$ and $\mathcal{O}_{\widetilde{Y}}$. ■

The vector bundle V is not trivial on the generic fibre. This issue will be addressed now by considering a specific twist $V' = V \otimes M$ with an appropriate line bundle M that makes V' trivial on a generic fibre.

As described in section 5 we analyse the situation near a point s of the base B that is a general point of an irreducible component Σ of the boundary divisor $S_{j,\mu}$. As explained there it suffices to consider the case where B is one-dimensional.

We thus consider a point $s \in B$ that is a general point of Σ and the fibre of $\widetilde{\mathbb{P}}$ over it. It is a chain

$$P_1, E_1, \ldots, E_{m-1}, P_2 \tag{3}$$

of smooth rational curves. The curve P_1 is a $\mathbb{P}^1$ with $j_1 = j$ or $b - j$ marked branch points and likewise P_2 is a copy of $\mathbb{P}^1$ with j_2 marked branch points with $j_1 + j_2 = b$, see the first paragraph of Section 4.

Recall that we have fixed a pair (j, μ) with $\mu = (m_1, \ldots, m_n)$. Assume now that $j_1 = b - j$ and $j_2 = j$. This choice will not affect our conclusion as we will see at the end of Section 8. We then have

$$b - j + d - n \equiv 0 \,(\mathrm{mod}\ 2) \qquad \text{and} \qquad j + d - n \equiv 0 \,(\mathrm{mod}\ 2)$$

because these are the degrees of ramification divisor of a curve over P_1 and P_2; moreover, the dual of the kernel of $\tilde{\pi}_* \mathcal{O}_{\widetilde{Y}} \to \mathcal{O}_{\widetilde{\mathbb{P}}}$ has degrees

$$(b - j + d - n)/2 \text{ on } P_1 \qquad \text{and} \qquad (j + d - n)/2 \text{ on } P_2\,.$$

We divide the latter degree by $d - 1$ and write r for the remainder

$$\frac{j + d - n}{2} = q(d-1) + r \qquad \text{with} \qquad 0 \le r < d - 1, \tag{4}$$

where of course q and r depend on j and d.

For uniformity of notation we rename the divisors in (3) by $R_i = R_i^{\Sigma}$ with

$$R_0 = P_1,\, R_1 = E_1, \ldots,\, R_{m-1} = E_{m-1},\, R_m = P_2\,.$$

We now twist V by a line bundle M such that $V \otimes M$ is trivial on the generic fibre. For this we could take as a first approximation the line bundle M corresponding to the divisor $-(k+1)\Xi_1$ with Ξ_1 the first section.

Now $V \otimes M$ is trivial on the generic fibre, hence as in equation (2) we have

$$c_1(V \otimes M) = p^* D + [A],$$

with D a divisor class on our base space and A a divisor supported on the singular fibre s. As explained in Section 3 we can get an effective class of the form $c_1(Q)+p^*c_1(R^1p_*V')+(-A)_{\mathrm{sh}}$ and for this we wish to minimize the expression $(-A)_{\mathrm{sh}}$. In other words, we want to adapt M such that we get a "good" A.

By Lemma 6.1 we know that $c_1(V)=W/2$ and by using the description of W given in Corollary 5.2, we find the degrees of the restriction of V on the various components R_i. This guides us to a reasonable choice of M and A.

We now add to $-(k+1)\Xi_1$ an integral linear combination of the curves from the chain $R_0, R_1, \ldots, R_{m-1}, R_m$ that has degree 0 on all the exceptional curves $E_i = R_i$ $(i=1,\ldots,m-1)$ such that the degree of V' is transferred as much as possible to $R_0 = P_1$. The expression

$$\sum_{i=0}^{m} i\,R_i \qquad \left(\text{respectively, } \sum_{i=0}^{m}(m-i)\,R_i\right)$$

has degree 0 on all E_i, degree 1 on P_1 and degree -1 on P_2 (respectively, degree -1 on P_1 and degree 1 on P_2).

We thus arrive at the standard line bundle M by which we will twist our vector bundle.

Definition 6.2. (Standard M.) We let M be the line bundle on $\widetilde{\mathbb{P}}$ associated to the divisor

$$-(k+1)\,\Xi_1 - \begin{cases} \sum_\Sigma q\,\sum_{i=0}^m (m-i)R_i^\Sigma & \text{if} \quad \Xi_1\cdot R_0^\Sigma \neq 0, \\ \sum_\Sigma (k+1-q)\sum_{i=0}^m i\,R_i^\Sigma & \text{if} \quad \Xi_1\cdot R_m^\Sigma \neq 0\,, \end{cases}$$

where the sum is taken over all irreducible boundary divisors Σ of $\widetilde{\mathcal{H}}_{d,g}$, $m=m(\mu)$, and q is given in (4). □

We summarize the result on the degrees of this vector bundle on the curves R_i.

Lemma 6.3. For given irreducible component of Σ of $S_{j,\mu}$ the degrees of the vector bundle $V' = V\otimes M$ on the curves $R_0^\Sigma,\ldots,R_m^\Sigma$ of the chain are: $d-n-r$ on R_0, r on R_m and $\deg W_E/2$ on R_i, that is, $(-d+2n-\sum_{\nu=1}^n d_{\nu,2})/2$ on R_1 and R_{m-1}, while on R_i for $2\leq i\leq m-2$ the degree is

$$\frac{1}{2}\sum_{\nu=1}^{n} -d_{\nu,i-1}+2d_{\nu,i}-d_{\nu,i+1}\,,$$

where as before $n = n(j, \mu)$ and $r = r(j, \mu)$ and $d_{\nu,i} = \gcd(m_\nu, i)$. The total degree of V' on the chain is equal to 0. □

Now we want to find a divisor A^Σ with support in the chain $R_0^\Sigma, \ldots, R_m^\Sigma$, that satisfies for a general point $s \in \Sigma$, the condition

$$\deg V' = \deg A^\Sigma$$

when restricted to the fibre over s.

In the following discussion we keep Σ and the pair (j, μ) fixed, therefore we allow ourselves occasionally to drop Σ from the notation. We write v_i for the degree of V' on a general fibre of R_i^Σ so that we have $\sum_{i=0}^m v_i = 0$. Our sought-for A^Σ will be given by

$$-A^\Sigma = \sum_{i=0}^{m-1} \alpha_i R_i^\Sigma ,$$

thus leaving out R_m^Σ. Starting with $\alpha_{m-1} = v_0 + \ldots + v_{m-1}$ gives the right degree on R_m and solving step by step via $\alpha_{m-i-1} = \alpha_{m-i} + \sum_{t=0}^{m-i-1} v_t$ we find the solution with α_i given by

$$\alpha_i = (m-i)\left(\sum_{t=0}^{m-1} v_t\right) - \sum_{t=i}^{m-1}(t-i)v_t .$$

We can extend the formula of Lemma 6.3

$$v_i = \frac{1}{2}\sum_{\nu=1}^{n} -d_{\nu,i-1} + 2\, d_{\nu,i} - d_{\nu,i+1}$$

for the degree of V' on a general fibre of $R_i = E_i$ for $i = 2, \ldots, m-1$ to $i = 0$ and $i = 1$ by putting formally

$$d_{\nu,0} = d_{\nu,m} = m_\nu, \qquad d_{\nu,-1} := \frac{2r}{n} + 1 .$$

Note that

$$-\sum_{t=i}^{m-1}(t-i)(-d_{\nu,t-1} + 2\, d_{\nu,t} - d_{\nu,t+1}) = (m-i-1)d_{\nu,m} - (m-i)d_{\nu,m-1} + d_{\nu,i}.$$

Then we find (using $\sum_{t=0}^{m-1} v_t = -v_m$) that

$$\begin{aligned}\alpha_i &= -r(m-i) + \frac{1}{2}\sum_{\nu=1}^{n}(d_{\nu,i} - (m-i)\, d_{\nu,m-1} + (m-i-1)\, d_{\nu,m}) \\ &= -r + \frac{1}{2}\sum_{\nu=1}^{n}(d_{\nu,i} - (m-i) + (m-i-1)\, m_\nu) \\ &= \frac{1}{2}\left((m-i)(d-n-2r) - d + \sum_{\nu=1}^{n} d_{\nu,i}\right).\end{aligned}$$

Definition 6.4. Now define $c = c(j,\mu)$ and $\delta_i = \delta_i(\mu)$

$$c := d - n - 2r \qquad \text{and} \qquad \delta_i := d - \sum_{\nu=1}^{n} d_{\nu,i} \quad \text{for } i = 0, \dots, m$$

with $d_{\nu,i}$ defined as $\gcd(m_\nu, i)$ for $i \geq 1$, by $d_{\nu,0} = m_\nu$ and $d_{\nu,-1} = 2r/n + 1$. □

Then we have $\delta_0 = 0$ and we can write $-A = \sum_{i=0}^{m-1} \alpha_i R_i$ with $\alpha_i = \frac{1}{2}((m-i)c - \delta_i)$. Observe that

$$c = \deg V'_{|P_1} - \deg V'_{|P_2}$$

and by Corollary 5.2

$$W_E = \sum_{i=1}^{m-1} \delta_i R_i \,.$$

Conclusion 6.5. For each irreducible component Σ of $S_{j,\mu}$ the divisor

$$A^\Sigma = -\sum_{i=0}^{m-1} \frac{1}{2}((m-i)c - \delta_i)\, R_i^\Sigma$$

has the property that the degree of A^Σ when restricted to a general fibre of R_i^Σ equals the degree of V' restricted to that fibre. □

For $\mathcal{L}$ trivial, $V = V_{\mathcal{L}}$ and with the standard choice for M as in Definition 6.2 we set

$$A_{\mathrm{st}} := \sum_{\Sigma} A^\Sigma \tag{5}$$

with A^{Σ} as in Conclusion 6.5 . Then the first Chern class $c_1(V')$ of $V' = V \otimes M$ satisfies the equality (2)

$$c_1(V') = p^*D + A_{\mathrm{st}}$$

with D a divisor class on the base.

7 Extensions of the Maroni Divisor

In this section we will work on the Hurwitz space $\overline{\mathcal{H}}_{d,g}$ and its normalization, where we assume as before that $g = (d-1)k$. In the preceding section we have constructed a vector bundle V of rank $d-1$ on $\widetilde{\mathcal{H}}_{d,g}$ and a twist $V' = V \otimes M$ by an explicit line bundle M such that V' is trivial on the generic fibre.

Now we define an effective divisor on $\widetilde{\mathcal{H}}_{d,g}$ by applying Proposition 3.8 and Theorem 3.10 of Section 3 to the bundle V'. This involves a divisor $-A$ given by $c_1(V') \equiv A$ modulo a pull back from the base and we make the result effective by adding F_A as in Definition 3.7 for the choice of A as in Conclusion 6.5.

Definition 7.1. (The standard extended Maroni divisor class $\mathfrak{m}_{\mathrm{st}}$.) Let $\mathcal{L}$ be trivial line bundle on $\widetilde{Y}$, $V = V_{\mathcal{L}}$, and choose M as in Definition 6.2 and put $V' = V \otimes M$. If Q denotes the degeneracy locus of the evaluation map $p^*p_*V' \to V'$ and with $A = A_{\mathrm{st}}$ as in (5) the class

$$c_1(Q) + p^*c_1(R^1p_*V') + (-A)_{\mathrm{sh}}$$

is effective and pulling it back via a section of $\widetilde{\mathbb{P}} \to \widetilde{\mathcal{H}}_{d,g}$ defines an effective divisor class $\mathfrak{m}_{\mathrm{st}}$ on $\widetilde{\mathcal{H}}_{d,g}$ which is the pull back under a section of p of the class

$$p^*p_*\left(\frac{1}{2}[A]\cdot\theta_p - \frac{1}{2r}[A]^2\right) + p^*p_*\left(c_2(V) - \frac{r-1}{2r}c_1^2(V)\right) + F_A\,,$$

and agrees with the Maroni locus on the open part $\mathcal{H}_{d,g}$. We call $\mathfrak{m}_{\mathrm{st}}$ the standard extended Maroni class. □

We refer to Definition 3.6 for the notation $(-A)_{\mathrm{sh}}$, to 3.7 for the notation F_A and Remark 3.9 for the meaning of F_A.

We can vary this definition by taking $\mathcal{L} = \mathcal{O}_{\widetilde{Y}}(Z)$ associated to an effective divisor Z with support in the boundary of $\widetilde{\mathcal{C}}$ and by twisting $V_{\mathcal{L}} \otimes M$ by a line bundle N with support in the boundary of $\widetilde{\mathbb{P}}$.

Definition 7.2. (The class $\mathfrak{m}_{\mathcal{L},N}$.) By taking $\mathcal{L}$ an effective line bundle $\mathcal{L} = \mathcal{O}_{\widetilde{Y}}(Z)$ as in Section 6, letting $V' = V'_{\mathcal{L},N} = V_{\mathcal{L}} \otimes M \otimes N$ and letting Q be the degeneracy locus of the evaluation map $p^* p_* V' \to V'$ and defining $A = A_{\mathcal{L},N}$ as in (2) the class

$$c_1(Q) + p^* c_1(R^1 p_* V') + (-A)_{\mathrm{sh}}$$

is effective and pulling it back via a section of $p : \widetilde{\mathbb{P}} \to \widetilde{\mathcal{H}}_{d,g}$ defines an effective divisor class $\mathfrak{m}_{\mathcal{L},N}$ on $\widetilde{\mathcal{H}}_{d,g}$ which by Theorem 3.10 is the pull back under a section of p of the class of the form

$$p^* p_* \left(\frac{1}{2} A \cdot \theta_p - \frac{1}{2r} A^2 \right) + p^* p_* \left(c_2(V) - \frac{r-1}{2r} c_1^2(V) \right) + F_A \,,$$

and agrees with the Maroni locus on the open part $\mathcal{H}_{d,g}$. □

For $\mathcal{L}$ and N trivial we have $\mathfrak{m}_{\mathcal{L},N} = \mathfrak{m}_{\mathrm{st}}$. In the next section we will work out the various terms in the formula for the basic case $\mathfrak{m}_{\mathrm{st}}$ of the standard extended Maroni class.

8 The Class of the Standard Extended Maroni Divisor

We now calculate the class $\mathfrak{m}_{\mathrm{st}}$ defined in the preceding section. We begin with the Chern classes in the formula of Definition 7.1. Recall the diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \widetilde{\mathcal{C}} \arrow[d, "c"] \\ \widetilde{\mathbb{P}} \arrow[r, "\beta"] & \mathbb{P} \end{tikzcd}$$

We also recall that by Lemma 6.1 we have $c_1(V) = W/2$. The branch divisor W of π was calculated in Corollary 5.2 and consists of two disjoint parts: the sum $W_S = \Xi$ of the b sections and a contribution $W_E = \sum_{i=1}^{m-1} \delta_i E_i$ from the exceptional divisors.

Now we move to the second Chern class. We have $c_2(V) = c_2(\pi_* \mathcal{O}_Y)$, so we consider $q_! \mathcal{O}_{\widetilde{Y}}$. On the one hand we have $q = \varpi \circ \rho$ and observe that $\rho_! \mathcal{O}_{\widetilde{Y}} = \rho_* \mathcal{O}_{\widetilde{Y}}$ because ρ blows down resolutions of rational singularities. Hence we get

$$q_! \mathcal{O}_{\widetilde{Y}} = \varpi_! (\rho_* \mathcal{O}_{\widetilde{Y}}) = \varpi_! \mathcal{O}_{\tilde{\mathcal{C}}} = \mathcal{O}_B - \mathbb{E}^\vee$$

with $\mathbb{E} = \varpi_* \omega_\varpi$, the Hodge bundle, by Serre duality the dual of $R^1 \varpi_* \mathcal{O}_{\tilde{\mathcal{C}}}$. We conclude that

$$c_1(q_! \mathcal{O}_{\widetilde{Y}}) = \lambda \,,$$

where λ denotes the Hodge class. On the other hand we have $q = p \circ \pi \circ \nu$ (see Diagram 4.2) and we have by Lemma 4.5 that $q_! \mathcal{O}_{\widetilde{Y}} = p_!(\pi_* \mathcal{O}_Y)$ and this can be calculated by applying Grothendieck–Riemann–Roch to the morphism p and the sheaf $\pi_* \mathcal{O}_Y$. This gives

$$\begin{aligned} c_1(p_!(\pi_*\mathcal{O}_Y)) =& p_*[\mathrm{ch}(\pi_*\mathcal{O}_Y)\,\mathrm{td}T_p]_{(2)} \\ =& p_*\left[d\,\mathrm{td}_2 T_p - \frac{1}{2}c_1(\pi_*\mathcal{O}_Y)\,\theta_p + \frac{c_1^2(\pi_*\mathcal{O}_Y) - 2\,c_2(\pi_*\mathcal{O}_Y)}{2}\right] \\ =& p_*\left[\frac{1}{4}W\theta_p + \frac{1}{8}W^2 - c_2(V)\right], \end{aligned}$$

where we used that $p_*(\mathrm{td}_2 T_p) = 0$ and $c_1(\pi_*\mathcal{O}_Y) = -W/2$, see Lemma 6.1. We thus see that

$$\lambda = c_1(q_!\mathcal{O}_{\widetilde{Y}}) = p_*\big(W\,\theta_p/4 + W^2/8 - c_2(V)\big)$$

and thus obtain a formula for $p_* c_2(V)$:

$$p_* c_2(V) = -\lambda + \frac{1}{8} p_*(W^2 + 2W\,\theta_p)\,.$$

We summarize.

Proposition 8.1. We have

$$p_* c_1^2(V) = \frac{1}{4} p_* W^2 \quad \text{and} \quad p_* c_2(V) = -\lambda + \frac{1}{8} p_*(W^2 + 2W\,\theta_p)\,. \qquad \square$$

From now on we shall work over a one-dimensional base B in the Hurwitz space. We need to justify that it suffices to prove an identity between divisor classes on $\widetilde{\mathcal{H}}_{d,g}$ by verifying it for all one-dimensional curves B in $\widetilde{\mathcal{H}}_{d,g}$ that intersect the boundary of our normalized compactified Hurwitz space $\widetilde{\mathcal{H}}_{d,g}$ transversally at general points. By repeatedly intersecting with generic hyperplane sections and applying the Lefschetz hyperplane theorem we reduce ourselves to a smooth surface S with the property that the Picard group of $\widetilde{\mathcal{H}}_{d,g}$ injects into $\mathrm{Pic}(S)$. The boundary of the Hurwitz space defines a configuration of transversally intersecting curves T on S and we assume that we have a line bundle L (representing our divisorial identity) on S which is trivial on every curve B that intersects T transversally at sufficiently general points. We wish to conclude that L is trivial. We claim that the Néron–Severi group of S is generated by the classes of such curves B. Indeed, the general hyperplane section H of S intersects T transversally at general points and hence L is trivial on it. For any curve E the class $E + mH$ contains for sufficiently large m a representative curve that intersects T as neatly as desired. Hence the first Chern class of L is orthogonal to all of $\mathrm{NS}(S)$, so by the non-degenerateness of the intersection form a non-zero multiple of it vanishes. Hence a non-zero integral multiple of it comes from a class in $H^1(S, \mathcal{O}_S)$ under the exponential map $H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^*)$. But by Kodaira and Spencer the map $H^1(S, \mathcal{O}_S) \to H^1(H, \mathcal{O}_H)$ is injective for a general hyperplane section of S, hence if L restricts trivially to H it must be trivial. Since we work in the rational Picard group this suffices.

If $m(\mu) > 1$ (recall that $m(\mu)$ is the l.c.m. of the m_i) then the self-intersection number W_E^2 is negative. Indeed, since the chain $E_1, \ldots, E_{m-1}$ can be contracted its intersection matrix is negative definite and since W_E is not the zero divisor the result follows. We can calculate this self-intersection number as follows. For each pair (j, μ) we have $W_E = \sum_{i=0}^{m} \delta_i E_i$ with $\delta_0 = \delta_m = 0$ (see Definition 6.4) and $W_E^2 = \sum_{i=1}^{m-1}(-2\delta_i^2 + 2\,\delta_i\delta_{i+1})$, in other words,

$$W_E^2 = -\sum_{i=1}^{m} (\delta_i - \delta_{i-1})^2 \, .$$

We have $W_E \cdot \theta_p = 0$ and thus $W \cdot \theta_p = \Xi \cdot \theta_p = \psi$ (with Ξ given in Corollary 5.2 and ψ the usual sum of the tautological classes ψ_i defined by the sections), which gives for each intersection point of B with $S_{j,\mu}$ a contribution $m(\mu)\, j(b-j)/(b-1)$, see formula (2) for ψ in [19]. Since $p_*(W_S^2) = -\psi$ we get from each intersection point of B with $S_{j,\mu}$ a contribution

$$-\frac{j(b-j)}{b-1} m(\mu) \, .$$

Now we move to the contributions of the divisor A. First recall the definition of A. For each irreducible component Σ of $S_{j,\mu}$ we have a contribution $-A^{\Sigma}$ with coefficients given by

$$\alpha_i = \frac{1}{2}((m-i)c - \delta_i)$$

and have for the contribution of the point $s \in B \cap S_{j,\mu}$ to the self-intersection number of A

$$\begin{aligned} -\alpha_0^2 + 2\sum_{i=1}^{m-1} \alpha_i(\alpha_{i-1} - \alpha_i) &= -\frac{m^2c^2}{4} + \frac{1}{2}\sum_{i=1}^{m-1}((m-i)c - \delta_i)(c - \delta_{i-1} + \delta_i) \\ &= -\frac{mc^2}{4} - \frac{1}{2}\sum_{i=1}^{m-1} \delta_i(\delta_i - \delta_{i-1}) \, , \end{aligned}$$

where we used $\sum_i (m-i)(\delta_{i-1}-\delta_i) = -\sum_i \delta_i$. Noting that $\delta_0 = \delta_m = 0$ and using again the identity

$$2\sum_{i=1}^{m-1} \delta_i(\delta_i - \delta_{i-1}) = \sum_{i=1}^{m} (\delta_i - \delta_{i-1})^2$$

we can rewrite this as

$$-\frac{mc^2}{4} - \frac{1}{4}\sum_{i=1}^{m} (\delta_{i-1} - \delta_i)^2 \, .$$

In general we will get a contribution to A from the resolution divisor of any intersection point s of B with $S_{j,\mu}$. This contribution depends only on the pair (j,μ) not on the chosen point. Therefore we get a contribution

$$-A = \sum_{j,\mu} (B \cdot S_{j,\mu}) \sum_{i=0}^{m(\mu)} \alpha_i(j,\mu)\, R_i(j,\mu) \, .$$

For the term $A \cdot \theta_p$ we get the negative of the coefficient of the "main" component R_0 which equals $m(\mu)\, c/2$. We thus arrive at the following lemma.

Lemma 8.2. The contribution of $S_{j,\mu}$ to the self-intersection number of A is for each intersection point s of $S_{j,\mu}$ with B equal to

$$-\frac{mc^2}{4} - \frac{1}{4}\sum_{i=1}^{m} (\delta_i - \delta_{i-1})^2 \, ,$$

while the contribution to $A \cdot \theta_p$ is $\frac{1}{2} m(\mu) c$. □

As to the term λ occurring in our formula of the extended Maroni class we recall the formula for the Hodge class λ from [13], which was reproved in an algebraic way in [19]. This formula says:

$$\lambda = \sum_{j=2}^{b/2} \sum_{\mu} m(\mu) \left(\frac{j(b-j)}{8(b-1)} - \frac{1}{12}\left(d - \sum_{\nu=1}^{n(\mu)} \frac{1}{m_\nu} \right) \right) S_{j,\mu} \, .$$

We have all the ingredients for the cycle class of the extended Maroni divisor.

Theorem 8.3. The class $\mathfrak{m}_{\mathrm{st}}$ of the standard extended Maroni divisor equals the effective class $\sum \sigma_{j,\mu} S_{j,\mu}$ with $\sigma_{j,\mu}$ given by

$$\sigma_{j,\mu} = m(\mu)\left(-\frac{|c_{j,\mu}|}{4} + \frac{c_{j,\mu}^2}{8(d-1)} + \frac{1}{12}\left(d - \sum_{\nu=1}^{n(\mu)} \frac{1}{m_\nu}\right) + \frac{j(b-j)(d-2)}{8(b-1)(d-1)}\right),$$

where $\mu = (m_1, \ldots, m_{n(\mu)})$ and $c_{j,\mu} = d - n - 2r$ with r the remainder of $(j + d - n)/2$ by division by $d - 1$. □

Proof. We use the definition of 7.1 and collect the various terms for a given pair (j, μ): the contribution of $p^* p_*(A \cdot \theta_p/2 - A^2/2(d-1))$ yields

$$\frac{1}{4} m(\mu) c_{j,\mu} + \frac{1}{8(d-1)}\left(m(\mu) c_{j,\mu}^2 + \sum_{i=1}^{m(\mu)} (\delta_i - \delta_{i-1})^2\right).$$

The contribution to F_A is given by the minimum coefficient: if $c_{j,\mu} \leq 0$ then the coefficients are ≥ 0 with the minimum corresponding to R_m which is 0; if $c_{j,\mu} > 0$ then the minimum is $-m c_{j,\mu}/2$.

The contribution of $-\lambda$ is

$$-m(\mu)\frac{j(b-j)}{8(b-1)} + m(\mu)\frac{1}{12}\left(d - \sum_\nu \frac{1}{m_\nu}\right)$$

and the contribution of $p_*(W \cdot \theta_p/4 + ([W_S]^2 + [W_E]^2)/8(d-1))$ is

$$\frac{j(b-j)}{4(b-1)} m(\mu) - \frac{j(b-j)}{8(d-1)(b-1)} m(\mu) - \frac{1}{8(d-1)} \sum_{i=1}^{m(\mu)} (\delta_i - \delta_{i-1})^2. \quad ■$$

In Theorem 8.3 the constant $c_{j,\mu}$ depends on the choice $j_2 = j$, the number of branch points on P_2. If we considered $j_2 = b - j$ instead, the role of j would be taken by $l = b - j$. Instead of $c = d - n - 2r$ we would get a constant $c' = d - n - 2r'$ with r' the remainder of $(l + d - n)/2$ by division by $d - 1$. But the formula is invariant under the change from c to c' as witnessed by the following lemma the proof of which is left to the reader.

Lemma 8.4. We have $|c|(|c| - 2(d-1)) = |c'|(|c'| - 2(d-1))$. □

9 Varying the Maroni Class by Twisting the Vector Bundle

The standard extended Maroni divisor $\mathfrak{m}_{\mathrm{st}}$ was obtained by starting with $\mathcal{L}$ trivial and taking $V = V_{\mathcal{L}}$ and then by tensoring with the standard M given in Definition 6.2 yielding a vector bundle V' trivial on the generic fibre. But we can replace V' by $V'_N = V' \otimes N$ with N any line bundle associated to a divisor supported on the singular fibres and preserve the property that the vector bundle is trivial on the generic fibre. In particular, we thus get rid of the special choice of the $k+1$ sections in Definition 6.2. We have $c_1(V_N) = p^*D + A_N$ with $A_N = A_{\mathrm{st}} + (d-1)N$. Using Definition 7.2 we get an effective class $\mathfrak{m}_{\mathcal{L},N}$ which is a pull back from $\widetilde{\mathcal{H}}_{d,g}$ and agrees with the Maroni locus on the open part $\mathcal{H}_{d,g}$. In this section we shall drop the trivial $\mathcal{L}$ from the notation and will write $\mathfrak{m}_N$ for this class. For N trivial we get back $\mathfrak{m}_{\mathrm{st}}$.

Again we may consider the situation locally as we did before. That is, we are working near an irreducible component Σ of the boundary divisor $S_{j,\mu}$ of the base $\widetilde{\mathcal{H}}_{d,g}$. But we may work on a one-dimensional base B and assume we are dealing with the multiplicity of one point s on B with fibre F that corresponds to the transversal intersection of B with the boundary divisor Σ of $\widetilde{\mathcal{H}}_{d,g}$ at a generic point of Σ.

We thus may write (abusing the notation slightly and omitting the index Σ) $N = \sum_{i=0}^{m} a_i R_i$ with $a_m = 0$ and similarly we write A instead of A^{Σ}.

We find that the difference $\mathfrak{m}_{\mathrm{st}} - \mathfrak{m}_N$ is the pull back under a section of p of

$$F_A - F_{A+(d-1)N} + p^* p_* \left(N \cdot A + \frac{d-1}{2} \left(N^2 - N \cdot \theta_p \right) \right),$$

and so $\mathrm{ord}_\Sigma \mathfrak{m}_{\mathrm{st}} - \mathrm{ord}_\Sigma \mathfrak{m}_N$ is the degree of the singular fibre F in

$$F_A - F_{A+(d-1)N} + f(N)\, F \tag{6}$$

with f the quadratic function

$$f(N) = N \cdot A + \frac{d-1}{2} \left(N^2 - N \cdot \theta_p \right).$$

If for some N we find that $\mathfrak{m}_{\mathrm{st}} - \mathfrak{m}_N$ is positive, we can improve the extended Maroni divisor by finding an effective divisor $\mathfrak{m}_N$ that contains the closure of the Maroni locus but is smaller than $\mathfrak{m}_{\mathrm{st}}$.

Discarding the term $F_A - F_{A+(d-1)N}$ in (6) we look for the maximum of the function f. The partial derivative of f with respect to N is $A + (d-1)(N - \theta_p/2)$ and this vanishes for

$$N_{\mathrm{crit}} = \frac{(d-1)\theta_p - 2A}{2(d-1)}.$$

Now θ_p near our fibre is represented by $-2S + \sum_{i=0}^{m-1} (m-i) R_i$, where S is a section that intersects $P_2 = R_m$. Using the form we found for A in Conclusion 6.5 we see that the coordinates of $N_{\rm crit} = \sum_{i=0}^{m-1} a_i R_i$ are given by

$$a_i = \frac{(m-i)a - \delta_i}{2(d-1)} \tag{7}$$

with $a := d-1+c$ and the function f assumes the value

$$\begin{aligned} f_{\max} = f(N_{\rm crit}) &= -\frac{d-1}{2} N_{\rm crit}^2 \\ &= m\left(\frac{c^2}{8(d-1)} + \frac{c}{4} + \frac{d-1}{8}\right) + \frac{1}{8(d-1)} \sum_{i=1}^{m} (\delta_{i-1} - \delta_i)^2 . \end{aligned}$$

The function f is bounded from above since the order of $\mathfrak{m}_N$ along Σ is non-negative, so the critical point must give a maximum. The function N^2 is negative semi-definite since N corresponds to a divisor supported on (singular) fibres. But the numbers given by (7) are not necessarily integral and the critical point $N_{\rm crit}$ lives in (the subgroup generated by the components of our fibre in) $\mathrm{Pic}(\widetilde{\mathbb{P}}) \otimes \mathbb{Q}$. Therefore we replace the rational number a_i given in (7) by a near integer α_i in the following way: first we choose $\alpha_{m-1} \in \mathbb{Z}$ such that

$$\alpha_{m-1} = a_{m-1} + e_{m-1} \qquad \text{with} \quad |e_{m-1}| \le 1/2$$

and then we choose successively the integer α_i for $i = m-2, \dots, 1$ such that

$$\alpha_i = a_i + e_i \qquad \text{with} \quad |e_i - e_{i+1}| \le 1/2 . \tag{8}$$

The function f, when viewed as a function of the coordinates of N, assumes at $\alpha = (\alpha_0, \dots, \alpha_{m-1})$ the value

$$f(\alpha) = f_{\max} - \frac{d-1}{2} \sum_{i=1}^{m} (e_{i-1} - e_i)^2 .$$

Lemma 9.1. The value of f at integral points takes its maximum at α. □

Proof. Let $h = (h_0, \dots, h_{m-1}) \in \mathbb{Z}^m$. Then we have

$$f(\alpha + h) = f(\alpha) - \frac{d-1}{2} \sum_{i=1}^{m} 2(e_{i-1} - e_i)(h_{i-1} - h_i) + (h_{i-1} - h_i)^2 ,$$

where we set $h_m = 0$. Writing $u_i = h_{i-1} - h_i$ and $\epsilon_i = e_{i-1} - e_i$, we see that $f(\alpha+h) - f(\alpha) = -((d-1)/2) \sum_{i=1}^{m} u_i^2 + 2\,\epsilon_i u_i$ and since $|\epsilon_i| \le 1/2$ by our choice of the e_i, this quadratic form in the u_i takes only non-positive values in integral points u_i. ■

Lemma 9.2. For $N = \sum_{i=0}^{m-1} \alpha_i R_i$ we have that $F_{A+(d-1)N} = 0$. $\square$

Proof. The divisor $A + (d-1)N$ has the form $\sum_{i=0}^{m-1} (d-1)[(m-i)/2 + e_i]R_i$. But $e_i \geq -(m-i)/2$, since we started with $e_{m-1} \geq -1/2$ and in each step we have $e_i \geq e_{i+1} - 1/2$ by (8). So all coefficients are ≥ 0 and the last one is zero. $\blacksquare$

Furthermore, we calculate the values of f at θ_p and the trivial line bundle and find that

$$f(\theta_p) = \frac{mc}{2} \qquad \text{and} \qquad f(0) = 0\,. \tag{9}$$

For $a_i = \alpha_i$ we have

$$(F_A, F_{A+(d-1)N}) = \begin{cases} (-\frac{mc}{2}F, 0) & c > 0, \\ (0,0) & c \leq 0, \end{cases} \tag{10}$$

with F denoting the full fibre. Lemma 9.1 together with (9) and (10) shows that if we write $F_A - F_{A+(d-1)N} = (f_A - f_{A+(d-1)N})F$ with F standing for a fibre we have

$$0 \leq f_A - f_{A+(d-1)N} + \begin{cases} f(\theta_p) & c > 0 \\ f(0,\ldots,0) & c \leq 0\,. \end{cases} \tag{11}$$

Collecting these facts we arrive at the following conclusion.

Theorem 9.3. Let Σ be an irreducible component of the boundary $S_{j,\mu}$ of $\overline{\mathcal{H}}_{d,g}$. Then the coefficient $\sigma_{j,\mu}$ of Σ of the locus $\cap_N \mathfrak{m}_N$ is equal to

$$m(\mu)\left(\frac{1}{12}(d - \sum_{\nu=1}^{n(\mu)} \frac{1}{m_\nu}) + \frac{j(b-j)(d-2)}{8(b-1)(d-1)}\right) - \frac{1}{8(d-1)} \sum_{i=1}^{m(\mu)} (\delta_{i-1} - \delta_i)^2$$
$$- \frac{d-1}{2}\left(\frac{m(\mu)}{4} - \sum_{i=1}^{m(\mu)} (e_{i-1} - e_i)^2\right).$$

$\square$

Proof. Let Σ be an irreducible component of the boundary $S_{j,\mu}$ of $\overline{\mathcal{H}}_{d,g}$. Then for the choice of $N = \sum_{i=0}^{m-1} \alpha_i R_i$ given by (8) we have $\mathfrak{m}_{\rm st} - \mathfrak{m}_N = \sigma\Sigma$ with σ given by

$$f_{\max} - \frac{d-1}{2} \sum_{i=1}^{m} (e_{i-1} - e_i)^2 + \begin{cases} mc/2 & c > 0 \\ 0 & c \leq 0\,. \end{cases} \tag{12}$$

Since this is non-negative by (9) and (11) the divisor class $\mathfrak{m}_N$ represents the class of an effective divisor containing the Maroni locus that is smaller by $\sigma\Sigma$ than the effective divisor representing $\mathfrak{m}_{\mathrm{st}}$ given in Theorem 8.3. ■

Remark 9.4. The correction term in the formula satisfies

$$\frac{m(\mu)}{4} - \sum_{i=1}^{m} (e_{i-1} - e_i)^2 \geq 0 . \qquad \square$$

The contribution σ in (12) depends only on the pair (j, μ) and for j only on the residue class mod $2(d-1)$ and it is a small correction. We list for $3 \leq d \leq 5$ the pairs $(j \,(\mathrm{mod}\; 2(d-1)), \mu)$ for which we find a positive correction $\sigma(j, \mu)$.

Table 1.

μ	[3]	[4]	[4]	[3, 1]	[2, 2]	[5]	[5]
j	0	1	5	0	0	0	2
σ	1	1	1	1	1	2	1
μ	[5]	[4, 1]	[4, 1]	[3, 2]	[3, 2]	[3, 1, 1]	[2, 2, 1]
j	6	1	7	1	7	0	0
σ	1	1	1	1	1	1	1

10 Improving the Maroni Class by Variation of $\mathcal{L}$ and Twisting

In this section we investigate the behaviour of $\mathfrak{m}_{\mathcal{L},N}$ when we also vary $\mathcal{L}$. We write $\mathcal{L} = \mathcal{O}_{\tilde{Y}}(Z)$ with Z an effective divisor supported on the boundary of $\tilde{Y}$.

In order to calculate the difference $\mathfrak{m}_{\mathrm{st}} - \mathfrak{m}_{\mathcal{L},N}$ we need to calculate some Chern classes. Recall that for the effective line bundle $\mathcal{L}$ the reflexive sheaf $V_{\mathcal{L}}$ is defined as the dual of the cokernel $K_{\mathcal{L}}$ of the natural map $\mathcal{O}_{\widetilde{\mathbb{P}}} \to \tilde{\pi}_* \mathcal{L}$. We fix the notation by

$$V := V_{\mathcal{O}_{\tilde{Y}}}, \quad V' = V \otimes M, \quad \text{and} \quad V_1' = V_{\mathcal{L}} \otimes M \otimes N,$$

where N corresponds to a divisor class supported on the boundary of $\widetilde{\mathbb{P}}$ and M is defined in 6.2. In case of $\mathcal{L} = \mathcal{O}_{\tilde{Y}}$ the results of Section 4 show that the direct image $\tilde{\pi}_* \mathcal{O}_{\tilde{Y}}$ is a vector bundle.

An application of Grothendieck–Riemann–Roch (to $\tilde{\pi}$, $\mathcal{L}$ and $\mathcal{O}_{\widetilde{\mathcal{O}}}$) allows us to compare the Chern classes of $\tilde{\pi}_* \mathcal{O}_{\widetilde{\mathbb{P}}}$ and $\tilde{\pi}_* \mathcal{L}$. We leave the proof to the reader.

Proposition 10.1. For $\mathcal{L} = \mathcal{O}_{\tilde{Y}}(Z)$ we have with U the ramification locus of $\tilde{\pi}$:

$$c_1(\tilde{\pi}_*\mathcal{L}) = \tilde{\pi}_*(Z) - \frac{1}{2}\tilde{\pi}_*(U),$$
$$c_2(\tilde{\pi}_*\mathcal{L}) - c_2(\pi_*\mathcal{O}_{\tilde{Y}}) = \frac{1}{2}((\tilde{\pi}_*Z)^2 - \tilde{\pi}_*Z^2) - \frac{1}{2}(\tilde{\pi}_*U\cdot\tilde{\pi}_*Z - \tilde{\pi}_*(U\cdot Z)).$$ □

Now we make the following assumption.

Assumption 10.2. We assume:

(1) The effective divisor Z on $\widetilde{Y}$ is supported on the boundary divisor and $\mathcal{L} = \mathcal{O}(Z)$ is the pull back of an effective line bundle on Y.
(2) The image of the Section 1 under $\iota_{\mathcal{L}} : \mathcal{O}_{\widetilde{\mathbb{P}}} \to \tilde{\pi}_*\mathcal{L}$ vanishes nowhere on $\widetilde{\mathbb{P}}$ □

By 10.2 (1) if $\mathcal{L} = \nu^*\mathcal{L}'$, the sheaf $\tilde{\pi}_*\mathcal{L} = \pi_*\mathcal{L}'$ is locally free since π is a finite flat morphism. The second assumption implies in particular that Z does not contain a divisor of the form $\tilde{\pi}^*D$ for an effective divisor D on $\widetilde{\mathbb{P}}$.

Proposition 10.3. Under the Assumption 10.2 we have $c_i((\pi_*\mathcal{L})^\vee) = c_i((K_{\mathcal{L}})^\vee)$ for $i = 1, 2$. □

Proof. This follows because $\tilde{\pi}_*\mathcal{L}$ is locally free and the image of ι defines a non-zero section, hence a locally free direct summand of rank 1. ■

By our assumption $\tilde{\pi}_*\mathcal{L}$ is locally free, hence Proposition 10.1 tells us the Chern classes of $\tilde{\pi}_*\mathcal{L}$ and hence of its dual. We thus find

$$c_2(V_{\mathcal{L}}\otimes N) - c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}) = \frac{1}{2}\left((\tilde{\pi}_*Z)^2 - \tilde{\pi}_*(Z^2)\right) - \frac{1}{2}(\tilde{\pi}_*U\cdot\tilde{\pi}_*U - \tilde{\pi}_*(U\cdot Z)) + (d-2)(-\tilde{\pi}_*Z + \frac{1}{2}\tilde{\pi}_*U)\cdot N + \binom{d-1}{2}N^2$$

and we observe that $\tilde{\pi}_*(U) = W$ and $c_1(V_1') - c_1(V') = (d-1)N - \tilde{\pi}_*Z$. We set

$$A_1 = A_{\rm st} + (d-1)N - \tilde{\pi}_*Z$$

and then have $c_1(V_1') = A_1 + p^*D$ for some divisor class D on the base as in (2).

If we calculate the expression $(1/2)A\cdot\theta_p - (1/(2(d-1))A^2$ for both V' and V_1' we find as difference

$$\frac{1}{2(d-1)}G\cdot(G + 2A - (d-1)\theta_p)\,,$$

where we write A for A_{st} and use

$$G := (d-1)N - \tilde{\pi}_* Z\,.$$

Now Theorem 3.10 gives the following result about the difference $\mathfrak{m}_{\mathrm{st}} - \mathfrak{m}_{\mathcal{L},N}$.

Theorem 10.4. Under Assumption 10.2 with $\mathcal{L} = \mathcal{O}_{\widetilde{Y}}(Z)$ the difference $\mathfrak{m}_{\mathrm{st}} - \mathfrak{m}_{\mathcal{L},N}$ is the pull back under a section of p of

$$F_A - F_{A_1} + \frac{1}{2}\tilde{\pi}_*(Z^2 - Z\cdot U) - \frac{1}{2(d-1)}\tilde{\pi}_* Z\cdot(\tilde{\pi}_* Z - \tilde{\pi}_* U) + \\ \frac{1}{2(d-1)} G\cdot(G + 2A - (d-1)\theta_p)\,,$$

with $\tilde{\pi}_* U = W$ and $A = A_{\mathrm{st}}$. □

This does not change if we add a full fibre of p to N.

If $\iota_{\mathcal{L}}(1)$ does not satisfy the condition of nowhere vanishing then $K_{\mathcal{L}}$ may not be torsion-free. For example, if $\mathcal{L}'$ is such that $\tilde{\pi}_*\mathcal{L}'$ is reflexive and if $\mathcal{L} = \mathcal{L}'\otimes\mathcal{O}(\tilde{\pi}^* D)$ for an effective divisor D on $\widetilde{\mathbb{P}}$ then by the projection formula we have $\tilde{\pi}_*\mathcal{L} = \tilde{\pi}_*\mathcal{L}'\otimes\mathcal{O}(D)$ and we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O} & \xrightarrow{=} & \mathcal{O} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{O}(D) & \to & \tilde{\pi}_*\mathcal{L} & \to & K_{\mathcal{L}'}\otimes O(D) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \\
0 & \to & \mathcal{O}_D(D) & \to & K_{\mathcal{L}} & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

showing that $O_D(D)$ is the torsion subsheaf of $K_{\mathcal{L}}$. Then the formula of Theorem 10.4 needs a correction due to the transition to the dual. In fact, if $Z_1 = Z + \tilde{\pi}^* D$, with Z satisfying the Assumption 10.2 and D effective then $\mathfrak{m}_{\mathcal{O}(Z_1),N} = \mathfrak{m}_{\mathcal{O}(Z),N(-D)}$.

We now wish to maximize this difference given in Theorem 10.4. Discarding the term $F_A - F_{A_1}$ we have a quadratic function f of Z and G which we can write as

$f = f_1(Z) + f_2(G)$ with

$$f_1 = \frac{1}{2}\tilde{\pi}_*(Z^2 - Z \cdot U) - \frac{1}{2(d-1)}\tilde{\pi}_* Z \cdot (\tilde{\pi}_* Z - \tilde{\pi}_* U)$$

and

$$f_2 = \frac{1}{2(d-1)} G \cdot (G + 2A - (d-1)\theta_p)$$

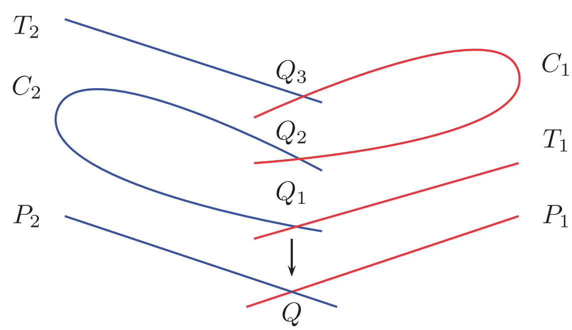
and we can maximize the two parts independently. The function f_2 is a quadratic function of G, a divisor class on $\widetilde{\mathbb{P}}$, while the function f_1 is a quadratic function of Z, a divisor class in the semi-group of the Picard group of $\widetilde{Y}$ generated by the effective divisors supported on the singular fibres. We can then consider the situation locally around a fibre of $\widetilde{Y}$ for f_1 and locally around a (corresponding) fibre of $\widetilde{\mathbb{P}}$ for f_2. The function f_2 (respectively, f_1) is then a quadratic function on the subgroup (respectively, semi-group) of the Picard group generated by the components of that fibre. For f_2 we observe that f_2 remains unchanged if we add to the divisor class of N a multiple of the fibre. Therefore we can consider the function f_2 on the subgroup generated by the components of the fibre minus one component. We leave out the component intersected by a section and since θ_p is represented by $-2S$ plus components from the fibre we can represent θ_p by a divisor with support on the fibre. Then we can find, similarly to what we did in Section 9, the maximum by looking for a critical point of the quadratic function by differentiating and get

$$G + A - \frac{d-1}{2}\theta_p = 0.$$

Then $G = -A + (d-1)\theta_p/2$ defines using the intersection form a functional on the subgroup of the Picard group generated by the components of the fibre and this functional can be represented by a rational linear combination of the fibre components, uniquely up to addition of a multiple of the fibre. Since it is a rational combination we can approximate it by an integral linear combination that is optimal. Since we know that the order of $\mathfrak{m}_{\mathcal{L},N}$ along any boundary component Σ is non-negative we see that the quadratic function f_2 is bounded from above and the critical point gives the maximum of f_2.

Next we look at the function f_1 as quadratic function (of Z) on the semi-group Γ generated by the components of the fibre, this time of $\widetilde{Y}$. By our assumption 10.2 (2) we have excluded the case that Z is the full preimage of an effective divisor on $\widetilde{\mathbb{P}}$. In fact, using such Z boils down to tensoring with a line bundle N on $\widetilde{\mathbb{P}}$ as we saw above. But we now also have the constraint that Z is effective. Therefore the method of using the critical point may not yield the maximum. The obvious critical point of f_1 given by $Z = U/2$ may not represent an effective divisor with support on the fibre and a critical point may not represent the maximum. In any case, either the critical point is effective and gives a maximum or the maximum is to be found on the boundary of the closure of the rational polyhedral cone generated by Γ in the $\mathbb{Q}$-vectorspace generated by Γ. Again, the fact that the order of $\mathfrak{m}_{\mathcal{L},N}$ is non-negative guarantees that f_1 has a maximum on the subset of the semi-group with rational coefficients generated by Γ that is allowed by our Assumption 10.2. The situation depends very much on the parity and divisibility properties of the (intersection) numbers involved. We give one example.

Example 10.5. We consider the trigonal case $d = 3$, g even, and an irreducible component Σ of $S_{j,[1,1,1]}$ whose generic point corresponds to the following picture



We have components C_i (respectively, T_i) mapping to P_i with degree 2 (respectively, degree 1) for $i = 1, 2$. The intersection behaviour is $C_i^2 = -2$, $T_i^2 = -1$ and $T_1T_2 = 0$, $C_1C_2 = 1$, $C_1T_2 = 1 = C_2T_1$, and $C_1T_1 = 0 = C_2T_2$. Given our conditions 10.2 and the symmetry we consider effective $\mathbb{Q}$-linear combinations $a_1C_1 + b_2T_2$. In the $\mathbb{Q}$-vectorspace $\langle C_1, T_2\rangle$ of such linear combinations we can represent U by the non-effective cycle $-lC_1 - lT_2$ since $UC_1 = l$ and $UT_2 = 0$. The quadratic form $(1/2)\tilde{\pi}_*(Z^2 - ZU) - (1/(2(d-1))(\tilde{\pi}_*(Z)(\tilde{\pi}_*Z - \tilde{\pi}_*U)$ takes the form $b_2(b - l - b_2)/4$ with critical value $(b-l)^2/16$ for $b_2 = (b-l)/2$. If we look on the $\mathbb{Q}$-space $\langle C_1\rangle$ we get the zero form while on the space $\langle T_1\rangle$ the quadratic form is $b_1(l - b_1)/4$ with critical point $b_1 = l/2$ giving the value $l^2/16$ and either this or $(b-l)^2/16$ represents indeed the effective maximum of the function f_1 in this case. □

In the next section we will work out one specific case where we can apply Theorem 10.4 and of which the above example is a special case.

11 A Special Case

We shall apply the preceding in the case where one of the ramification indices m_ν equals 1. We shall assume $d \geq 3$. The results will be used in the Section 13.

So we look again locally on our one-dimensional base B near a point s of an irreducible divisor Σ of the boundary $S_{j,\mu}$ of our Hurwitz space and we want to calculate

$$\mathrm{ord}_\Sigma \mathfrak{m}_{\mathrm{st}} - \mathrm{ord}_\Sigma \mathfrak{m}_{\mathcal{L},N}$$

for a suitable choice of divisors Z and N.

We suppose that μ is such that one of the indices μ_ν equals 1. If the ramification index $m_\nu = 1$ for a point Q_ν on the fibre of $\tilde{C}$ over s then locally near the preimage under the map $\widetilde{Y} \to \mathcal{C}$ of the point Q_ν (see the beginning of Section 4) our space Y is smooth according to Lemma 5.1. The fibre of p over s is a chain of rational curves $R_0 = P_1, R_1 = E_1, \dots, R_{m-1} = E_{m-1}, R_m = P_2$ and above it in Y and $\widetilde{Y}$ we have a corresponding chain $T_0, \dots, T_m$ such that $T_i \to R_i$ is unramified of degree 1 for $i = 1, \dots, m-1$ because of $\mu_\nu = 1$; we further require that T_0 is a rational tail and $T_0 \to R_0$ is unramified of degree 1. The resolution map $\nu : \widetilde{Y} \to Y$ does not affect the local situation and hence the proper transforms of the T_i do not intersect the exceptional locus of ν. We now choose

$$N = \sum_{i=0}^{m-1} a_i R_i \quad \text{and} \quad Z = \sum_{i=0}^{m-1} b_i T_i \quad \text{so that} \quad \tilde{\pi}_* Z = \sum_{i=0}^{m-1} b_i R_i\,,$$

where the coefficients b_i are non-negative.

We have $T_0^2 = R_0^2 = -1$ while $T_j^2 = R_j^2 = -2$ for $1 \leq j \leq m-1$; Moreover, since the maps $T_i \to R_i$ ($i = 0, \dots, m-1$) are unramified of degree 1 we have $Z \cdot U = 0$ and by the projection formula we have $\tilde{\pi}_*(Z^2) = (\tilde{\pi}_* Z)^2$. Thus we see that the contribution of Theorem 10.4 is $f_A - f_{A_1} + f(Z,N)$ with $f(Z,N)$ the quadratic expression

$$f(Z,N) := \frac{1}{2(d-1)} \left((d-2)X^2 + WX + G(G - (d-1)\theta_p + 2A) \right) ,$$

where we used $X = \tilde{\pi}_* Z$ and $W = \tilde{\pi}_*(U)$. Discarding again $F_A - F_{A_1}$, we look for the critical point of the function $f(Z,N)$ viewed as a function of the variables X and G. The vanishing of the partial derivatives to X and G is given by

$$2(d-2)\,X + W = 0 \qquad \text{and} \qquad 2\,G - (d-1)\theta_p + 2\,A = 0\,.$$

So the critical point is given by

$$(X, G) = (-W/2(d-2), (d-1)\theta_p/2 - A)\,.$$

This means that this pair (X, G) defines on the subgroup of $\mathrm{Pic}(\widetilde{\mathbb{P}})$ generated by the components R_i ($i = 0, \ldots, m-1$) of our fibre the same intersection behaviour as the pair of critical point $(\tilde{\pi}_* Z, (d-1)N - \tilde{\pi}_* Z)$ we are looking for. The value at the critical point is given by

$$-\frac{d-1}{8}\theta_p^2 + \frac{1}{2}A\theta_p - \frac{1}{2(d-1)}A^2 - \frac{1}{8(d-1)(d-2)}W^2\,.$$

Observe that near our fibre $W = \Xi + \sum_{i=0}^{m-1} \delta_i R_i$ with Ξ the sum of the sections, and therefore in the subgroup of $\widetilde{\mathbb{P}}$ generated by $R_0, \ldots, R_{m-1}$ we have $W = -\sum_{i=1}^{m-1}[(m-i)l - \delta_i]R_i$. Moreover $A = -(1/2)\sum_{i=0}^{m-1}((m-i)c - \delta_i)R_i$ and $\theta_p = \sum_{i=0}^{m-1}(m-i)R_i - 2S$ with S the section that intersects R_m only. The coordinates of the critical point in terms of the coordinates g_i of G and x_i of X (for $i = 0, \ldots, m-1$) are

$$g_i = \frac{(m-i)a - \delta_i}{2} \qquad \text{and} \qquad x_i = \frac{(m-i)l - \delta_i}{2(d-2)} \qquad \text{for } i = 0, \ldots, m-1\,,$$

with $a = c + d - 1$. Observe that the x_i are non-negative. The value taken by f at the critical point is

$$f_{\max} = \frac{m(\mu)}{8(d-1)(d-2)}\left((b-j)^2 + (d-2)a^2\right) + \frac{1}{8(d-2)}\sum_{i=1}^{m(\mu)}(\delta_{i-1} - \delta_i)^2\,. \qquad (14)$$

We let $N = \sum_{i=1}^{m-1} a_i R_i$. We then have $(d-1)a_i = g_i + x_i$. Since the a_i and x_i are not necessarily integers we define integers α_i and ξ_i by

$$\alpha_i = a_i + e_i, \qquad \xi_i = x_i + e_i' \quad \text{for } i = 0, \ldots, m-1\,, \qquad (15)$$

where the e_i and e_i' are chosen as follows. We choose $e_{m-1} \in [-1/2, 1/2)$ and successively we choose e_i for $i = m-2, \ldots, 1$ such that $|e_i - e_{i+1}| \leq 1/2$. By noting that $a - l = (d-1)(2(q-k-1)+1)$, with $k = g/(d-1)$ and q given in (4), we observe that

$$a_i - x_i = \frac{(m-i)(a-l)}{2(d-1)} = \frac{(m-i)(2(q-k-1)+1)}{2}\,,$$

which is an integer or a half-integer according to the parity of $m - i$. We then choose the e_i' such that $e_i' = e_i$ for $m-i$ even and $e_i' = e_i + 1/2$ for $m-i$ odd. The value of f at the point $(\alpha_0,\dots,\alpha_{m-1},\xi_0,\dots,\xi_{m-1})$ is $f_{\max}+f_{\rm err}$ with $f_{\rm err}$ given by

$$-\frac{d-1}{2}\sum_{i=1}^{m}(e_{i-1}-e_i)^2+\sum_{i=1}^{m}(e_{i-1}-e_i)(e'_{i-1}-e'_i)-\frac{1}{2}\sum_{i=1}^{m}(e'_{i-1}-e'_i)^2$$

which simplifies to

$$-\frac{d-2}{2}\sum_{i=1}^{m}(e_{i-1}-e_i)^2-\frac{1}{2}\sum_{i=1}^{m}((e_{i-1}-e_i)-(e'_{i-1}-e'_i))^2\,, \tag{16}$$

so that we get

$$f_{\rm err}=-\frac{d-2}{2}\sum_{i=1}^{m}(e_{i-1}-e_i)^2-\frac{m}{8}\,. \tag{17}$$

Lemma 11.1. The maximum of f in integral points is taken at (α,ξ). □

Proof. For any choice of rational numbers e_i, e'_i and with α_i and ξ_i as in (15) we have $f(\alpha,\xi)=f_{\max}+f_{\rm err}$ with $f_{\rm err}$ as in (16). We have to maximize $f_{\rm err}$. Using $t_i=e_{i-1}-e_i$ and $s_i=e'_{i-1}-e'_i$ we can rewrite it as

$$-\frac{d-2}{2}\sum_{i=1}^{m}t_i^2-\frac{1}{2}\sum_{i=1}^{m}(t_i-s_i)^2\,.$$

The term $\sum_{i=1}^{m}(t_i-s_i)^2$ is minimized independently of the choice of the e_i by $m/8$ by choosing $e_i=e'_i$ or $e_i=e'_i-1/2$ depending on the parity of $m-i$. Then by the argument of Lemma 9.1 $f_{\rm err}$ is maximized by our choice of the e_i. ■

Lemma 11.2. For (α,ξ) as defined above we have $f_{A+G}=0$. □

Proof. We have

$$A+G=\sum_{i=0}^{m-1}\left(\frac{(m-i)(d-1)}{2}+(d-2)e_i-\epsilon_i\right)R_i$$

with $\epsilon_i=0$ if $m-i$ is even and $\epsilon_i=1/2$ otherwise. Since $(d-2)e_i-\epsilon_i\geq-(d-1)/2$, the coefficients are non-negative while the coefficient of R_m is zero. ■

Theorem 11.3. Let Σ be an irreducible component of the boundary $S_{j,\mu}$ of $\widetilde{\mathcal{H}}_{d,g}$ corresponding to the case (j,μ) with at least one m_ν equal to 1. We also assume that $T_0 \to R_0$ is unramified of degree 1. Then the coefficient $\sigma_{j,\mu}$ of Σ of the locus $\cap_{\mathcal{L},N}\mathfrak{m}_{\mathcal{L},N}$ is equal to

$$m(\mu)\left(\frac{1}{12}\left(d-\sum_{\nu=1}^{n(\mu)}\frac{1}{m_\nu}\right)+\frac{j(b-j)(d-2)}{8(b-1)(d-1)}\right)-\frac{1}{8(d-2)}\sum_{i=1}^{m(\mu)}(\delta_{i-1}-\delta_i)^2$$
$$-\frac{m(\mu)(b-j)^2}{8(d-1)(d-2)}-\frac{d-2}{2}\left(\frac{m(\mu)}{4}-\sum_{i=1}^{m(\mu)}(e_{i-1}-e_i)^2\right). \qquad \square$$

Proof. By choosing N and $\mathcal{L}$ corresponding to the near critical points (α,ξ) given in (15) we see that we can diminish the coefficient of Σ in $\mathfrak{m}_{\mathrm{st}}$ by $f_A - f_{A_1} + f_{\max} + f_{\mathrm{err}}$. For the term $f_A - f_{A_1}$ the identity (10) remains valid by Lemma 11.2. Also the positivity argument given there can be used: we have $f((d-1)\theta_p, 0) = mc/2$ and $f(0,0) = 0$. Collecting all the terms gives the result. ■

The same method can also be applied in more complicated situations. For example, we may assume that $\mathbb{P}$ has a singularity at $Q = P_1 \cap P_2$ of the type $s^m = uv$ and that over it in the fibre of Y at Q_ν we have a singularity of type $s^{m/m_\nu} = x_\nu y_\nu$. Then on $\widetilde{\mathbb{P}}$ we find a chain $R_0 = P_1, R_1, \ldots, R_{m-1}, R_m = P_2$ and on Y we find a chain with $T_{\nu,i}$ with $i = 1,\ldots,m-1$ with $T_{\nu,i}$ mapping to R_i with degree $d_{\nu,i} = \gcd(m_\nu, i)$ and ramification $m_\nu/d_{\nu,i}$. Let C_ν be the component lying over R_0 containing the point Q_ν and assume for simplicity that it contains no other singular points. Then $\pi^*(R_i) = \sum_\nu T'_{\nu,i}$ with $T'_{\nu,i} = (m_\nu/d_{\nu,i})T_{\nu,i}$ a Cartier divisor. We can then apply intersection theory on the normal surface Y (e.g., we have the projection formula for $\pi : Y \to \tilde{\mathbb{P}}$ and $\nu : \widetilde{Y} \to Y$; see [6], p. 867) and one calculates

$${T'_{\nu,i}}^2 = -2m_\nu, \quad C_\nu T'_{\nu,i} = T'_{\nu,i}T'_{\nu,i+1} = m_\nu\,.$$

If we now put $Z_1 = a_{\nu,0}C_\nu + \sum_{i=1}^{m-1} a_{\nu,i}T'_{\nu,i}$ and set $Z = \nu^* Z_1$ then these are effective Cartier divisors and we can calculate

$$Z^2 = -m_\nu \sum_{i=1}^{m}(a_{\nu,i-1}-a_{\nu,i})^2, \quad \nu^* T'_{\nu,i} U = -d_{\nu,i-1} + 2d_{\nu,i} - d_{\nu,i+1}\,;$$

moreover we have

$$\nu^* C_\nu \cdot U = 3(m_\nu - 1) + 2g(C_\nu)$$

together with

$$(\tilde{\pi}_* Z)^2 = -\sum_{i=1}^{m} (d_{\nu,i-1} a_{\nu,i-1} - d_{\nu,i} a_{\nu,i})^2$$

and

$$\tilde{\pi}_* Z \cdot W = a_{\nu,0} m_\nu l - \sum_{i=1}^{m} (d_{\nu,i-1} a_{\nu,i-1} - d_{\nu,i} a_{\nu,i})(\delta_{i-1} - \delta_i)\,.$$

With these formulas, similar in spirit to the above but more involved, we can calculate the extended Maroni classes in such cases using Theorem 10.4.

12 A Question

To a pair $(\mathcal{L}, N)$ with $\mathcal{L} = \mathcal{O}_{\widetilde{Y}}(Z)$ for an effective divisor Z supported on the boundary of $\widetilde{Y}$ and N a line bundle supported on the boundary of $\widetilde{\mathbb{P}}$ we associated the class of an effective divisor $\mathfrak{m}_{\mathcal{L},N}$ on $\overline{\mathcal{H}}_{d,g}$ containing the closure of the Maroni divisor. By varying $\mathcal{L}$ and N we can define an effective divisor class

$$\mathfrak{m}_{\min} = \cap_{\mathcal{L},N} \mathfrak{m}_{\mathcal{L},N}\,.$$

This is the minimal class we can get with this method. The formulas we found for the extended Maroni classes $\mathfrak{m}_{\mathcal{L},N}$ are given in terms of the boundary classes $S_{j,m}$. The rational Picard group of $\overline{\mathcal{H}}_{d,g}$ is not known in general but is conjectured to be generated by the boundary classes. In his thesis [15, 16] Patel proves that the boundary classes on the unordered Hurwitz space are independent. If the classes of the irreducible boundary components Σ are linearly independent we may describe this minimal Maroni class $\mathfrak{m}_{\min}$ by

$$\mathfrak{m}_{\min} := \sum_{\Sigma} \alpha_\Sigma \Sigma$$

with α_Σ given by

$$\alpha_\Sigma = \min\{\alpha_\Sigma(\mathcal{L}, N) : \mathcal{L}, N\}\,,$$

where we write

$$\mathfrak{m}_{\mathcal{L},N} = \sum_{\Sigma} \alpha_\Sigma(\mathcal{L}, N)\Sigma\,,$$

and with $\mathcal{L} = \mathcal{O}_{\widetilde{Y}}(Z)$ ranging over the line bundles associated to effective divisor classes with support on the boundary and N over the line bundles corresponding to divisor classes with support on the boundary. The following natural question comes up.

Question 12.1. Does the class $\mathfrak{m}_{\min}$ coincide with the Zariski closure of the Maroni locus? □

In the next section we show that in the trigonal case the answer is positive.

13 Comparison with the Results of Patel and Deopurkar–Patel

In this section we compare our results with the results of Patel [15] on a partial compactification and with the results of Deopurkar–Patel [5] for the trigonal case.

We begin by comparing with the results of Patel by specializing our calculation to a partial compactification of our Hurwitz space involving only the boundary divisors $S_{j,\mu}$ of the Hurwitz space $\overline{\mathcal{H}}_{d,g}$ that parametrize covers that have $b-2$ ramification points on one $\mathbb{P}^1$ and 2 on the other; in other words $j = 2$. We then can compare our result with the results of Patel [15] on such a compactification. The boundary components $S_{2,\mu}$ involved fall into three cases:

$$\sum_{\mu} S_{2,\mu} = E_2 + E_3 + \Delta,$$

where E_2 collects the divisors corresponding to $\mu = (2, 2, 1, \ldots, 1)$, while E_3 collects those corresponding to $\mu = (3, 1, \ldots, 1)$ and the divisor Δ contains all $S_{2,\mu}$ with $\mu = (1, \ldots, 1)$. Each generic point of the divisor E_2 corresponds to curves with a g^1_d with two simple ramification points in one fibre, while each generic point of E_3 to those with a triple ramification point. Each irreducible component of the divisors E_2 and E_3 maps dominantly to a cycle in $\overline{\mathcal{M}}_g$ that intersects the interior $\mathcal{M}_g$. The divisor Δ includes the case of covers with a base point (i.e., the divisor of covers $C' \cup L \to \mathbb{P}^1$ with $C' \to \mathbb{P}^1$ of degree $d-1$ and $L \to \mathbb{P}^1$ of degree 1 and $L \cap C'$ one point, the "base point") but also the one-nodal curves.

We calculate the contributions to these terms in Theorem 8.3 using the identity $b = 2(k+1)(d-1)$:

$$-\frac{(k+1)(d-2)}{2(b-1)}\,\Delta + \frac{2k+1}{2(b-1)}\,E_2 - \frac{(d-10)(k+1)+4}{6\,(b-1)}\,E_3\,.$$

Except for a factor 2 this expression coincides with the expression that Patel finds in his thesis for the class of an effective divisor containing the Maroni locus on a partially compactified Hurwitz space that takes into account only the case where $j = 2$. The factor 2 is due to the automorphism interchanging the two points on one tail $\mathbb{P}^1$.

But as we shall show now, this is not the class of the closure of the Maroni locus. To show this, we consider the case where $\mu = (1, \ldots, 1)$. Then $m(\mu) = 1$ and our space $\widetilde{C}$ is smooth here. There are many irreducible components of the divisor $S_{j,\mu}$ for $j = 2$ and $\mu = (1, \ldots, 1)$. We further assume that we have $2g - 4 + 2d$ branch points on P_1 and 2 on P_2. One divisor contained in $S_{2,\mu}$ parametrizes covers where there are two irreducible curves C_1 and C_2 of genus g_1 and g_2 covering P_1 with degrees d_1 and d_2 with $d_1 + d_2 = d$ and these are joined by a rational tail mapping with degree 2 to P_2. Over an appropriate one-dimensional base B we have by the projection formula $C_i^2 = -d_i$. We now choose $\mathcal{L}$ associated to the effective divisor $Z = y\,C_1$ and $N = x\,P_1$ for suitable integers x and y with $y \geq 0$. Then we apply Proposition 10.4 and using that $C_1 \cdot U = 2(g_1 + d_1 - 1)$ we find the following.

Lemma 13.1. If we write $F_A - F_{A_1} = (f_A - f_{A_1})F_\Sigma$ with F_Σ the fibre over Σ we have that $\mathrm{ord}_\Sigma \mathfrak{m}_{\mathrm{st}} - \mathrm{ord}_\Sigma \mathfrak{m}_{\mathcal{L},N}$ is equal to

$$(f_A - f_{A_1}) + \left(x + k - \frac{y+1}{2}\right) y\, d_1 - \frac{1}{2} x(x-1)\,(d-1) + (1-g_1)y - x\,. \qquad \square$$

In certain cases this is positive. For example, if one takes $g_1 = 1$ where this parametrizes curves with an elliptic tail. Then we set $y = k$ and $x = 0$. We have $A_1 = P_1 - \tilde{\pi}^* Z = P_1 - kd_1 P_1$ and $F_A = 0$ and also $f_{A_1} = 1 - kd_1$. Then the above contribution is $k(k+1)d_1/2 - 1$ and this is positive. Therefore the divisor class found by Patel is larger than that of the Zariski closure of the Maroni locus.

Next we compare with the results of Deopurkar and Patel for the trigonal case. In this case the boundary divisors of $\overline{\mathcal{H}}_{3,g}$ are listed in [5, p. 872]. We have the following divisors: Δ, corresponding to a generic irreducible trigonal curve with a node, H, corresponding to the generic hyperelliptic curve, and divisors $\Delta_i(g_1, g_2)$ for $i = 1, \ldots, 6$. We refer to the description given in loc. cit. In that paper the formula

$$\text{Residual} = (7g+6)\lambda - g\delta - 2(g-3)\mu$$

(where μ refers to the Zariski closure of the Maroni locus and δ is the class of the pull back of the total boundary of $\overline{\mathcal{M}}_g$) given on page 877 can be used to calculate the coefficients of the boundary divisors in the expression for the closure of the Maroni locus. For example, take the case of $\Delta_1(g_1, g_2)$ with $g_1 + g_2 = g - 2$. The number of branch points on the base

P_1 (respectively, P_2) is $2(g_1+2)$ (respectively, $2(g-g_1)$) and the pull back of δ is $3\,\Delta_1$. If g_1 is even their formula gives

$$(7g+6)\lambda - 3g - 2(g-3)\mu = \frac{3}{2}g_1(g-g_1-2)$$

and using the contribution $(g_1+2)(g-g_1)/2(2g+3)$ to the Hodge class λ we find the contribution of the Maroni closure $\mu = \lambda/2$. Comparing with our formula for the standard Maroni extension $\mathfrak{m}_{\rm st}$ given in Theorem 8.3 we get $r=0$, $c=0$, hence $\sigma=\lambda/2$ and the results agree. Similarly, if g_1 is odd they find $\mu=\lambda/2-1/4$ and our Theorem 8.3 gives with $r=1$, $d-n=0$, and $c=-2$ that $\sigma=\lambda/2-1/4$. So in the case of the boundary divisors of type Δ_1 the two results agree. However, for some boundary divisors the result of Theorem 8.3 for the standard class $\mathfrak{m}_{\rm st}$ gives a higher multiplicity than the result of Deopurkar and Patel. We list these cases in a table.

Table 2.

Div	(g_1, g_2)	Difference
Δ		0
Δ_1		0
Δ_2		0
Δ_3	$g_1\equiv 0(\mathrm{mod}\,2)$	0
Δ_3	$g_1\equiv 1(\mathrm{mod}\,2)$	1
Δ_4	$g_2\equiv 0(\mathrm{mod}\,2)$	$(g_2+1)^2/4-1/4$
Δ_4	$g_2\equiv 1(\mathrm{mod}\,2)$	$(g_2+1)^2/4$
Δ_5		$g_2(g_2+1)/2$
Δ_6	$g_2\equiv 0(\mathrm{mod}\,2)$	$(g_2+1)^2/4-1/4$
Δ_6	$g_2\equiv 1(\mathrm{mod}\,2)$	$(g_2+1)^2/4$
H		$g(g+2)/4$

But here the methods given in Sections 9 and 10, or more specifically 11, that produce Maroni classes $\mathfrak{m}_{\mathcal{L},N}$, come to the rescue and by using this we retrieve the result of Deopurkar and Patel for the trigonal case as we now illustrate.

The first case where our divisor $\mathfrak{m}_{\rm st}$ is not optimal is the case of Δ_3 and g_1 odd where we get an additional coefficient 1 for $\Delta_3(g_1,g_2)$. Here $m=3$, $r=1$, and $c=0$, $(\delta_0,\delta_1,\delta_2,\delta_3)=(0,2,2,0)$, and $A=E_1+E_2$. But then we apply the results of Section 9 and take $N=P_1$; we see that the maximum is assumed in the point $(n_0,n_1,n_2,n_3)=(3/2,1/2,0,0)$ and we get according to the formula of Theorem 9.3 that $\sigma=1$, see Table 1 (with $j=2g_1+2\equiv 0(\mathrm{mod}\,4)$ and $\mu=[3]$). This gives the desired correction.

In the remaining cases $\Delta_4, \Delta_5, \Delta_6$, and H we apply the method of Section 11 and get as reduction of the coefficient $\sigma_{j,\mu}$ a quadratic polynomial in $b - j$ and in all these cases this is exactly equal to the difference in Table 2. We illustrate this by the case Δ_4. We have $g_1 + g_2 = g - 1$ and $\mu = (1, 1, 1)$ so that $d = n = 3$ and $m = 1$; moreover $j = 2(g - g_2 + 1)$ and $l = 2(g_2 + 1)$. Substituting this in Theorem 11.3 and comparing with the standard class $\mathfrak{m}_{\rm st}$ gives the correction term $(g_2+1)^2/4$ or $g_2(g_2+2)/4$ as desired.

The cases Δ_5 and Δ_6 follow similarly and the case H is the same as Δ_6 with $g_2 = g$ even.

Conclusion 13.2. In the trigonal case ($d = 3$ and g even) by choosing for each Σ the line bundles $\mathcal{L}$ and N as in Theorems 9.3 and 11.3 we get an effective divisor $\mathfrak{m}_{\mathcal{L},N}$ on $\overline{\mathcal{H}}_{3,g}$ that equals the Zariski closure of the Maroni locus on $\mathcal{H}_{3,g}$. □

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